The eigenvalues of the Laplacian on a sphere with boundary conditions specified on a segment of a great circle

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We prove that the eigenvalues of the Laplacian on a sphere with a Dirichlet boundary condition specified on a segment of a great circle lie between an integer and a half-integer and for a Neumann boundary condition they lie between a half integer and an integer. These eigenvalues correspond to the eigenvalues of the angular part of the Laplacian with boundary conditions specified on a plane angular sector, which are relevant in the calculation of scattering amplitude. These eigenvalues can also be used to determine the behavior of the fields near the tip of a plane angular sector as a function of the distance to the tip. The first few eigenvalues for both Dirichlet and Neumann boundary conditions are calculated. The same eigenvalues are also calculated using the Wentzel–Kramers–Brillouin (WKB) method. There is excellent agreement between the exact and the WKB eigenvalues. © *1997 American Institute of Physics.* [S0022-2488(97)00603-8]

I. INTRODUCTION

The problem of scattering of waves by an elliptic cone was first studied by Kraus and Levine.[1] They introduced the sphero–conal coordinate system in which the wave equation, satisfying boundary conditions on the surface of an elliptic cone, is separable. In this coordinate system the wave equation separates into two angular Lamé equations and the spherical Bessel equation. The solution of the wave equation for a plane angular sector (PAS) is a special case of the solution of the wave equation for an elliptic cone, because, as is shown in Fig. 1, a PAS is a degenerate elliptic cone. In the work by Kraus and Levine[1] a formal expression for the Green's function in terms of an eigenfunction expansion of the products of Lamé and spherical Bessel functions is derived, but no numerical results are reported.

Since the work of Kraus and Levine other authors have studied this problem, mainly concentrating on the scattering from a PAS. Radlow[2] studied the scattering of a plane wave from a quarter plane. He determined a two-variable integral representation of the field and then using a generalization of the Weiner–Hopf method, found a transformation that forces the field to zero on the quarter plane. Blume and Kirchner[3] studied the singular behavior of the field near the corner of a plane angular aperture and calculated the lowest eigenvalues for several different slot angles. Satterwhite[4] investigated the scattering of electromagnetic waves from a perfectly conducting plane angular sector. He expressed the scattered electric field in terms of an integral equation of the products of the dyadic Green's function and the surface current density. The dyadic Green's function was found as sums of products of vector wave functions whose components can be expressed in terms of the solutions of scalar wave equations. Satterwhite calculated the first few eigenvalues and eigenfunctions for the special case of a quarter plane, but did not report any results for the solutions of the scattered electric and magnetic fields. De Smedt and Van Bladel[5] also studied the singular behavior of the electric and magnetic fields near the tip of a PAS. They showed that the electric field is singular as $r^{\nu-1}$ and the magnetic field is singular as $r^{\tau-1}$, where r is the distance to the tip of the sector. They calculated the lowest values for ν and τ using a

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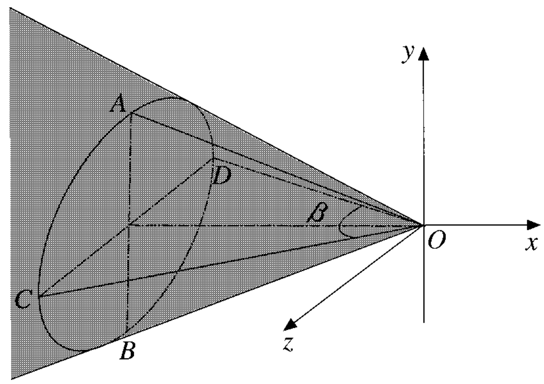

FIG. 1. This figure shows an elliptic cone with apex at the origin, which in the spheroconal coordinate system is represented by $\vartheta=\vartheta_0$, where ϑ_0 is the angle between OA and the positive x-axis. For $\vartheta_0=\pi$ the elliptic cone becomes degenerate (the elliptic base collapses to its major axis, CD) resulting in the plane angular sector, COD, with corner angle β. Note that $\beta=0$ corresponds to a needle and $\beta=\pi$ corresponds to a half-plane.

variational technique. These values of ν and τ respectively correspond to the lowest Dirichlet and Neumann eigenvalues discussed in this paper. Boersma[6] used the Babinet's principle to show that the electric singularity exponent for a conducting PAS is identical to the magnetic singularity exponent for the complementary PAS.

In Sec. II, we use the results of Kraus and Levine[1] to prove a theorem on the range of the eigenvalues of a PAS; and another theorem to prove that for a Dirichlet boundary condition these eigenvalues lie between an integer and a half-integer and for a Neumann boundary condition they lie between a half-integer and an integer. The first few eigenvalues for corner angles, $\beta=60°$, 90°, and 120°, are tabulated for both Dirichlet and Neumann boundary conditions. In Sec. III, the Wentzel–Kramers–Brillouin (WKB) solution of this problem is outlined and the same eigenvalues calculated by the WKB method are tabulated. The WKB eigenvalues which show remarkable agreement with the exact eigenvalues, also exhibit the same properties as do the exact eigenvalues, namely those stated by the two theorems in this paper. The details of the WKB treatment, including the calculation of the WKB eigenfunctions and normalization, is the topic of a subsequent paper. A method for calculating the exact eigenfunctions is included in the Appendix.

II. THE EXACT SOLUTION OF THE WAVE EQUATION FOR A PLANE ANGULAR SECTOR

The sphero–conal variables (ϑ,φ,r) are related to (x,y,z) by

$$x=r\cos\vartheta\sqrt{1-\kappa'^2\cos^2\varphi},$$

$$y=r\sin\vartheta\sin\varphi, \tag{1}$$

$$z=r\cos\varphi\sqrt{1-\kappa^2\cos^2\vartheta},$$

where $\kappa=\cos(\beta/2)$ and $\kappa'=\sqrt{1-\kappa^2}$; the range of the variables are

$$0\leqslant\vartheta\leqslant\pi,\quad 0\leqslant\varphi\leqslant2\pi,\quad r\geqslant0.$$

The construction of this coordinate system is described and its orthogonality proved in Ref. 1. The geometry of the coordinate system may briefly be described as follows: The coordinate r is the distance to the origin, so the surface $r=r_1$ is a sphere centered at the origin. The coordinate $\vartheta=\vartheta_1$ is a semi-infinite elliptic cone whose cross section in a plane $x=$constant is an ellipse centered on

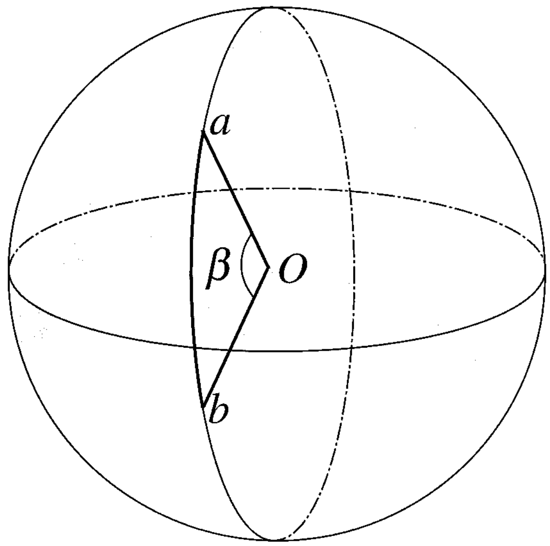

FIG. 2. The segment ab along a great circle.

the x-axis, with its major axis in the plane $y=0$. The surface $\varphi=\varphi_1$ is a semi-infinite elliptic half-cone whose cross section in a plane $z=$constant is half an ellipse centered on the z-axis with its major axis in the plane $y=0$. The coordinate system defined by Eq. (1) reduces to the spherical coordinate system when $\kappa=1$. For $\kappa\neq 1$ the coordinate surfaces $\vartheta=0$ and $\vartheta=\pi$ are plane angular sectors in the plane $y=0$.

The wave equation in the sphero–conal coordinate system can be separated into a radial equation[1]

$$\frac{d}{dr}\left(r^2\frac{d}{dr}R\right)+[k^2r^2-\nu(\nu+1)]R=0,$$

where the separation constant has been written as $\nu(\nu+1)$, and an angular equation

$$\Delta_\Omega V(\vartheta,\varphi)+\nu(\nu+1)V(\vartheta,\varphi)=0, \tag{2}$$

where the angular part of the Laplacian, Δ_Ω, is given by

$$\Delta_\Omega=\frac{1}{\kappa^2\sin^2\vartheta+\kappa'^2\sin^2\varphi}\left\{\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{\partial}{\partial\vartheta}\left(\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{\partial}{\partial\vartheta}\right)\right.$$
$$\left.+\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{\partial}{\partial\varphi}\left(\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{\partial}{\partial\varphi}\right)\right\}.$$

Mathematically, specifying boundary con ditions on a plane angular sector is equivalent to specifying boundary conditions on a segment of a great circle of a sphere on which Δ_Ω operates, see Fig. 2. If no boundary condition on the surface of the sphere is specified, the eigenvalues of Δ_Ω are integers and they correspond to the free space eigenvalues. If boundary conditions are specified on a great circle which extends from the north to the south pole, the eigenvalues of Δ_Ω are half-integers and they correspond to the eigenvalues of Δ_Ω for a half-plane. If, on the other hand, boundary conditions are specified along an arbitrary segment of a great circle, the eigenvalues correspond to the eigenvalues of Δ_Ω for a plane angular sector with corner angle β. By setting

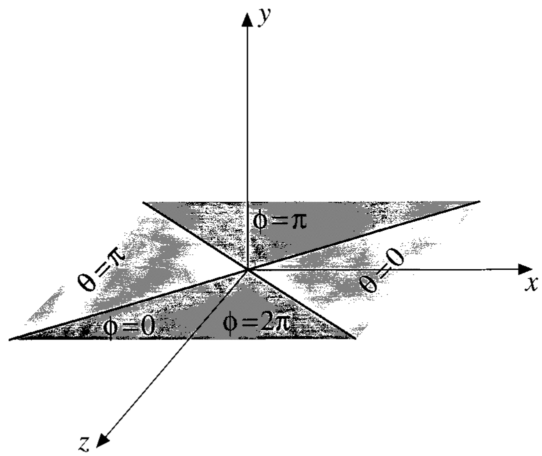

FIG. 3. The plane angular sectors $\vartheta=0$, $\vartheta=\pi$; $\varphi=0$, $\varphi=\pi$ and $\varphi=2\pi$.

$$V(\vartheta,\varphi)=\Theta(\vartheta)\Phi(\varphi),$$

the angular part can be separated into

$$\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\left[\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\,\Theta(\vartheta)\right]+[\nu(\nu+1)\kappa^2\sin^2\vartheta+\mu]\Theta(\vartheta)=0, \tag{3}$$

and

$$\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{d}{d\varphi}\left[\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{d}{d\varphi}\Phi(\varphi)\right]+[\nu(\nu+1)\kappa'^2\sin^2\varphi-\mu]\Phi(\varphi)=0, \tag{4}$$

where μ is another separation constant. The radial equation is the spherical Bessel equation and Eqs. (3) and (4) are the trigonometric Lamé differential equations.

The solution of the Laplace equation satisfying Dirichlet or Neumann boundary condition on the surface of a plane angular sector is of the form

$$\Psi(\vartheta,\varphi,r)=r^pV(\vartheta,\varphi),$$

where r is measured from the tip of the plane angular sector and the boundary surface, $\vartheta=\pi$, is shown in Fig. 3. When substituted in the Laplace equation,

$$\frac{d}{dr}\left(r^2\,\frac{d}{dr}\right)\Psi(\vartheta,\varphi,r)+\Delta_\Omega\Psi(\vartheta,\varphi,r)=0,$$

it gives

$$p(p+1)-\nu(\nu+1)=0,$$

where in obtaining the above results Eq. (2) has been used. The solutions of the above equation are $p=\nu$, and $p=-\nu-1$. Near the tip of the plane angular sector (r small), the physically possible solution is $p=\nu$, ($\nu>0$), then

$$\Psi(\vartheta,\varphi,r)=r^\nu V(\vartheta,\varphi),$$

which gives the r dependence of the potential near the tip of a plane angular sector. Thus, for a given boundary condition, the values of ν, which depend on the corner angle, can be used to determine the behavior of the fields and surface charge densities near the tip of a plane angular sector.

Eigenvalues of the exact solution. We take the boundary surface to be the sector $\vartheta=\pi$. The coordinate-imposed boundary condition on $\Phi(\varphi)$ is that it must be periodic with period 2π $\Phi(\varphi+2\pi)=\Phi(\varphi)$, in order to ensure that it is single-valued. If $\Phi(\varphi)$ is even, i.e., $\partial\Phi(\varphi)/\partial\varphi|_{\varphi=0} \equiv \Phi_e'(0)=0$, we can write

$$\Phi_e(\varphi+2\pi)=\Phi_e(\varphi)=\Phi_e(-\varphi),$$

or

$$\Phi_e'(\varphi+2\pi)=-\Phi_e'(-\varphi).$$

This implies

$$\Phi_e'(\pi)=0.$$

On the other hand, if $\Phi(\varphi)$ is odd, $\Phi_o(0)=0$ and

$$\Phi_o(\varphi+2\pi)=\Phi_o(\varphi)=-\Phi_o(-\varphi),$$

which implies

$$\Phi_o(\pi)=0.$$

Thus for the even and odd periodic cases we must respectively have

$$\Phi_e'(0)=\Phi_e'(\pi)=0,$$

and

$$\Phi_o(0)=\Phi_o(\pi)=0.$$

The boundary conditions on $\Theta(\vartheta)$ can be any of the following.

(1) **The even Dirichlet boundary condition**: In this case $\Theta(\vartheta)$ is even $(\Theta_e'(0)=0)$ and it satisfies the Dirichlet boundary condition on the boundary surface $(\Theta_e(\pi)=0)$. It has been shown by Kraus and Levine[1] that the factors $\Theta(\vartheta)$ and $\Phi(\vartheta)$ of the eigenfunction $V(\vartheta,\varphi)$ can only be both even or both odd. Since $\Theta(\vartheta)$ has been chosen to be even, $\Phi(\varphi)$ must also be even resulting in the following boundary conditions:

$$\begin{cases} \Theta_e'(0)=0, & \Theta_e(\pi)=0, \\ \Phi_e'(0)=0, & \Phi_e'(\pi)=0. \end{cases} \tag{5}$$

(2) **The odd Neumann boundary condition**: In this case $\Theta(\vartheta)$ is odd $(\Theta_o(0)=0)$ and it satisfies the Neumann boundary condition on the boundary surface $\Theta_o'(\pi)=0$. Then $\Phi(\varphi)$ must also be odd resulting in the following boundary conditions:

$$\begin{cases} \Theta_o(0)=0, & \Theta_o'(\pi)=0, \\ \Phi_o(0)=0, & \Phi_o(\pi)=0. \end{cases} \tag{6}$$

(3) **The odd Dirichlet and the even Neumann boundary conditions**: By using the above arguments, for the odd Dirichlet case we have

$$\begin{cases} \Theta_o(0)=0, & \Theta_o(\pi)=0, \\ \Phi_o(0)=0, & \Phi_o(\pi)=0. \end{cases}$$

By writing

$$\Theta_o(\vartheta)=\Theta(\vartheta)-\Theta(-\vartheta),$$

and imposing the boundary condition $\Theta_o(\pi)=0$, we find

$$\Theta(-\pi)=\Theta(\pi). \tag{7}$$

Similarly, for the even Neumann boundary condition we have

$$\begin{cases} \Theta_e'(0)=0, & \Theta_e'(\pi)=0, \\ \Phi_e'(0)=0, & \Phi_e'(\pi)=0 \end{cases}.$$

In this case

$$\Theta_e(\vartheta)=\Theta(\vartheta)+\Theta(-\vartheta),$$

or

$$\Theta_e'(\vartheta)=\Theta'(\vartheta)-\Theta'(-\vartheta).$$

At the boundary surface the left-hand side of the second equation in the above vanishes, resulting in

$$\Theta'(\pi)=\Theta'(-\pi). \tag{8}$$

For the odd Dirichlet and the even Neumann boundary conditions both $\Phi(\varphi)$ and $\Theta(\vartheta)$ are periodic with period 2π which results in integer eigenvalues. Furthermore, Eqs. (7) and (8) suggest that $\Theta(\vartheta)$ is continuous across the boundary surface, which is the case when no boundary surface is present. The odd Dirichlet and even Neumann boundary conditions therefore are the same as the free space boundary conditions. The first few eigenvalues of Δ_Ω satisfying the odd Dirichlet, the even Neumann, and the half-plane boundary conditions on a PAS are tabulated in the Appendix. In this paper we are only interested in the even Dirichlet and odd Neumann solutions, because these solutions correspond to the case when a boundary surface is present. From this point on we drop the ‘‘*e*’’ and ‘‘*o*’’ subscripts and refer to the even Dirichlet and odd Neumann cases as the Dirichlet and Neumann cases, respectively.

The eigenvalues ν and μ for the Dirichlet and Neumann boundary conditions are obtained by simultaneously solving Eqs. (3) and (4) and imposing the appropriate boundary conditions given by Eqs. (5) and (6).

Theorem 1: For a given value of ν, μ can only take values satisfying

$$\nu(\nu+1)\kappa'^2 \geqslant \mu \geqslant -\nu(\nu+1)\kappa^2.$$

Proof: Equation (3) can be written as

$$\frac{d}{d\vartheta}\left[\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\,\Theta(\vartheta)\right]=-\frac{\nu(\nu+1)\kappa^2\sin^2\vartheta+\mu}{\sqrt{1-\kappa^2\cos^2\vartheta}}\,\Theta(\vartheta).$$

Multiplying both sides of the above equation by $\Theta(\vartheta)$ and integrating from 0 to π gives

$$\int_0^\pi \Theta(\vartheta) \frac{d}{d\vartheta}\left[\sqrt{1-\kappa^2 \cos^2 \vartheta} \frac{d}{d\vartheta} \Theta(\vartheta)\right] d\vartheta = -\int_0^\pi \frac{\nu(\nu+1)\kappa^2 \sin^2 \vartheta + \mu}{\sqrt{1-\kappa^2 \cos^2 \vartheta}} \Theta^2(\vartheta) d\vartheta.$$

The left-hand side can be integrated by parts to yield

$$\Theta(\vartheta) \frac{d}{d\vartheta} \Theta(\vartheta) \sqrt{1-\kappa^2 \cos^2 \vartheta}\Big|_0^\pi - \int_0^\pi \left(\frac{d}{d\vartheta} \Theta(\vartheta) \right)^2 \sqrt{1-\kappa^2 \cos^2 \vartheta} d\vartheta$$

$$= -\int_0^\pi \frac{\nu(\nu+1)\kappa^2 \sin^2 \vartheta + \mu}{\sqrt{1-\kappa^2 \cos^2 \vartheta}} \Theta^2(\vartheta) d\vartheta.$$

The first term is zero for all boundary conditions, so we are left with

$$\int_0^\pi \left(\frac{d}{d\vartheta} \Theta(\vartheta) \right)^2 \sqrt{1-\kappa^2 \cos^2 \vartheta} d\vartheta = \int_0^\pi \frac{\nu(\nu+1)\kappa^2 \sin^2 \vartheta + \mu}{\sqrt{1-\kappa^2 \cos^2 \vartheta}} \Theta^2(\vartheta) d\vartheta.$$

The left-hand side of the above equation is positive, so we must have

$$\nu(\nu+1)\kappa^2 \int_0^\pi \frac{\sin^2 \vartheta \Theta^2(\vartheta)}{\sqrt{1-\kappa^2 \cos^2 \vartheta}} d\vartheta + \mu \int_0^\pi \frac{\Theta^2(\vartheta)}{\sqrt{1-\kappa^2 \cos^2 \vartheta}} d\vartheta \geqslant 0.$$

Let

$$I_1 = \int_0^\pi \frac{\sin^2 \vartheta \Theta^2(\vartheta)}{\sqrt{1-\kappa^2 \cos^2 \vartheta}} d\vartheta,$$

and

$$I_2 = \int_0^\pi \frac{\Theta^2(\vartheta)}{\sqrt{1-\kappa^2 \cos^2 \vartheta}} d\vartheta,$$

then

$$\nu(\nu+1)\kappa^2 I_1 + \mu I_2 \geqslant 0 \Rightarrow \mu \geqslant -\frac{I_1}{I_2} \nu(\nu+1)\kappa^2.$$

Since

$$1 \geqslant \frac{I_1}{I_2} \geqslant 0,$$

the smallest possible value that μ can take is when $I_1/I_2 = 1$, i.e.,

$$\mu \geqslant -\nu(\nu+1)\kappa^2.$$

From Eq. (4) we get

$$\int_0^\pi \left(\frac{d}{d\varphi} \Phi(\varphi) \right)^2 \sqrt{1-\kappa'^2 \cos^2 \varphi} d\varphi = \int_0^\pi \frac{\nu(\nu+1)\kappa'^2 \sin^2 \varphi - \mu}{\sqrt{1-\kappa'^2 \cos^2 \varphi}} \Phi^2(\varphi) d\varphi,$$

which yields

$$\nu(\nu+1)\kappa'^2 J_1 - \mu J_2 \geqslant 0 \Rightarrow \mu \geqslant \frac{J_1}{J_2}\,\nu(\nu+1)\kappa'^2,$$

where J_1 and J_2 are I_1 and I_2 with κ replaced by κ', ϑ replaced by φ, and $\Theta(\vartheta)$ replaced by $\Phi(\varphi)$. Here also

$$1 \geqslant \frac{J_1}{J_2} \geqslant 0,$$

we therefore get

$$\mu \leqslant \nu(\nu+1)\kappa'^2.$$

From the two relations for μ we can write

$$\nu(\nu+1)\kappa'^2 \geqslant \mu \geqslant -\nu(\nu+1)\kappa^2. \tag{9}$$

□

Theorem 2: For any non-negative integer n, the eigenvalues of Δ_Ω with the Dirichlet boundary condition specified on a PAS with corner angle $0<\beta<\pi$ (segment of a great circle) satisfy:

$$n<\nu<n+\tfrac{1}{2},$$

and for the Neumann boundary condition they satisfy:

$$n+\tfrac{1}{2}<\nu<n+1.$$

Proof: The proof of this theorem is based on the fact that all positive integers are eigenvalues of Δ_Ω for the free space boundary condition and all positive half-integers are eigenvalues of Δ_Ω for the half-plane boundary condition. The proof will be carried out in three parts. First, we use the variational principle on the Θ equation [Eq. (3)] to prove that the eigenvalue, ν, corresponding to Dirichlet boundary condition on a PAS is larger than some non-negative integer q. Then we use the variational principle on the Φ equation [Eq. (4)] to prove that this same eigenvalue is smaller than $q'+1/2$, where q' is some other non-negative integer. Finally, we show that $q'=q$, completing the proof. Consider the Θ and Φ equations,

$$\frac{d}{d\vartheta}\left[\sqrt{1-\kappa^2\cos^2\vartheta}\,\Theta'(\vartheta)\right]+\frac{\nu(\nu+1)\kappa^2\sin^2\vartheta+\mu}{\sqrt{1-\kappa^2\cos^2\vartheta}}\,\Theta(\vartheta)=0, \tag{10}$$

$$\frac{d}{d\varphi}\left[\sqrt{1-\kappa'^2\cos^2\varphi}\,\Phi'(\varphi)\right]+\frac{\nu(\nu+1)\kappa'^2\sin^2\varphi-\mu}{\sqrt{1-\kappa'^2\cos^2\varphi}}\,\Phi(\varphi)=0. \tag{11}$$

Equations (10) and (11) are the Euler–Lagrange equations for the functionals

$$\int_0^\pi\left[\sqrt{1-\kappa^2\cos^2\vartheta}\,\Theta'^2(\vartheta)-\frac{\mu}{\sqrt{1-\kappa^2\cos^2\vartheta}}\,\Theta^2(\vartheta)\right]d\vartheta,$$

and

$$\int_0^\pi \left[\sqrt{1-\kappa'^2 \cos^2 \varphi}\,\Phi'^2(\varphi) + \frac{\mu}{\sqrt{1-\kappa'^2 \cos^2 \varphi}}\,\Phi^2(\varphi) \right] d\varphi.$$

The eigenvalues of these equations, $\nu(\nu+1)\equiv\alpha$, are then the stationary values of the functionals[7]

$$[\alpha] = \frac{\int_0^\pi \left[\sqrt{1-\kappa^2 \cos^2 \vartheta}\,\Theta'^2(\vartheta) - \frac{\mu}{\sqrt{1-\kappa^2 \cos^2 \vartheta}}\,\Theta^2(\vartheta) \right] d\vartheta}{\int_0^\pi \frac{\kappa^2 \sin^2 \vartheta}{\sqrt{1-\kappa^2 \cos^2 \vartheta}}\,\Theta^2(\vartheta)\, d\vartheta}, \tag{12}$$

and

$$[\alpha] = \frac{\int_0^\pi \left[\sqrt{1-\kappa'^2 \cos^2 \varphi}\,\Phi'^2(\varphi) + \frac{\mu}{\sqrt{1-\kappa'^2 \cos^2 \varphi}}\,\Phi^2(\varphi) \right] d\varphi}{\int_0^\pi \frac{\kappa'^2 \sin^2 \varphi}{\sqrt{1-\kappa'^2 \cos^2 \varphi}}\,\Phi^2(\varphi)\, d\varphi}, \tag{13}$$

where Θ and Φ satisfy some type of boundary conditions. The free space boundary condition in the sphero–conal coordinate system are given by

$$\Theta'(0)=0, \quad \Theta'(\pi)=0, \quad \Phi'(0)=0, \quad \Phi'(\pi)=0. \tag{14}$$

Note that the above boundary conditions also correspond to the even Neumann boundary condition for a PAS. The Dirichlet boundary condition for a PAS is

$$\Theta'(0)=0, \quad \Theta(\pi)=0, \quad \Phi'(0)=0, \quad \Phi'(\pi)=0, \tag{15}$$

and the Dirichlet boundary condition for a half-plane in the sphero–conal coordinate system is (In this case the half-plane is made up of two plane angular sectors, $\vartheta=\pi$ and $\varphi=\pi$, see Fig. 3)

$$\Theta'(0)=0, \quad \Theta(\pi)=0, \quad \Phi'(0)=0, \quad \Phi(\pi)=0. \tag{16}$$

Recall that the solution of Eqs. (3) and (4) subject to boundary conditions (14) or (16), respectively, correspond to integer or half-integer values of the eigenvalue, ν. By comparing the above boundary conditions, we note that the boundary condition for a PAS can be obtained from the free space boundary condition by changing the boundary condition on the Θ equation from $\Theta'(\pi)=0$ to $\Theta(\pi)=0$, and leaving the boundary condition for the Φ equation unchanged. Similarly, the boundary condition for a half-plane can be obtained from the boundary condition for a PAS by changing the boundary condition on the Φ equation from $\Phi'(\pi)=0$ to $\Phi(\pi)=0$, and leaving the boundary condition for the Θ equation unchanged. We can relate the free space boundary condition to the PAS boundary condition by defining a function f in the following way:

$$\Theta'(0)=0, \quad \Theta'(\pi)=-f\,\frac{\Theta(\pi)}{\kappa'}, \quad \Phi'(0)=0, \quad \Phi'(\pi)=0. \tag{17}$$

In this case the free space boundary condition corresponds to $f=0$, and the PAS boundary condition corresponds to $f=\infty$. The eigenvalues, in this case, are the stationary values of the functional

$$[\alpha]=\frac{\int_0^{\pi}\left[\sqrt{1-\kappa^2\cos^2\vartheta}\,\Theta'^2(\vartheta)-\frac{\mu}{\sqrt{1-\kappa^2\cos^2\vartheta}}\Theta^2(\vartheta)\right]d\vartheta+f\Theta^2(\pi)}{\int_0^{\pi}\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2\vartheta}}\Theta^2(\vartheta)d\vartheta},\tag{18}$$

where the variations $\delta\Theta(0)$ and $\delta\Theta(\pi)$ are unrestricted. Furthermore, we find

$$\frac{\delta\alpha}{\delta f}=\frac{\Theta^2(\pi)}{\left[\int_0^{\pi}\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2\vartheta}}\Theta^2(\vartheta)d\vartheta+\frac{\delta\mu}{\delta\alpha}\int_0^{\pi}\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}\Theta^2(\vartheta)d\vartheta\right]}.$$

In order to guarantee that $\delta\alpha/\delta f>0$, it is sufficient that $\delta\mu/\delta\alpha$ be positive. From Eq. (13) we find that

$$\frac{\delta\alpha}{\delta\mu}=\frac{\int_0^{\pi}\frac{1}{\sqrt{1-\kappa'^2\cos^2\varphi}}\Phi^2(\varphi)d\varphi}{\int_0^{\pi}\frac{\kappa'^2\sin^2(\varphi)}{\sqrt{1-\kappa'^2\cos^2\varphi}}\Phi^2(\varphi)d\varphi}>0,$$

thus $\delta\alpha/\delta f>0$. This implies that as f increases from 0, which corresponds to free space boundary condition, to ∞, which corresponds to the boundary condition on a PAS, the corresponding eigenvalue also increases. Here if the free space eigenvalue reached as $f\to 0$ has the value q, we must have $\nu>q$. This completes the first part of the proof.

Next we want to relate the PAS boundary condition to the half-plane boundary condition by defining

$$\Theta'(0)=0,\quad \Theta(\pi)=0,\quad \Phi'(0)=0,\quad \Phi'(\pi)=-g\,\frac{\Phi(\pi)}{\kappa}.\tag{19}$$

Now the PAS boundary condition corresponds to $g=0$ and the half-plane boundary condition corresponds to $g=\infty$. The eigenvalues are then the stationary values of the functional

$$[\alpha]=\frac{\int_0^{\pi}\left[\sqrt{1-\kappa'^2\cos^2\varphi}\,\Phi'^2(\varphi)+\frac{\mu}{\sqrt{1-\kappa'^2\cos^2\varphi}}\Phi^2(\varphi)\right]d\varphi+g\Phi^2(\pi)}{\int_0^{\pi}\frac{\kappa'^2\sin^2\varphi}{\sqrt{1-\kappa'^2\cos^2\varphi}}\Phi^2(\varphi)d\varphi},\tag{20}$$

where now the variation $\delta\Phi(0)=0$, but $\delta\Phi(\pi)$ is unrestricted. We find

$$\frac{\delta\alpha}{\delta g}=\frac{\Phi^2(\pi)}{\left[\int_0^{\pi}\frac{\kappa'^2\sin^2\varphi}{\sqrt{1-\kappa'^2\cos^2\varphi}}\Phi^2(\varphi)d\varphi-\frac{\delta\mu}{\delta\alpha}\int_0^{\pi}\frac{1}{\sqrt{1-\kappa'^2\cos^2\varphi}}\Phi^2(\varphi)d\varphi\right]}.$$

$\delta\mu/\delta\alpha$ must be negative to guarantee that $\delta\alpha/\delta g>0$. From Eq. (12) we find that

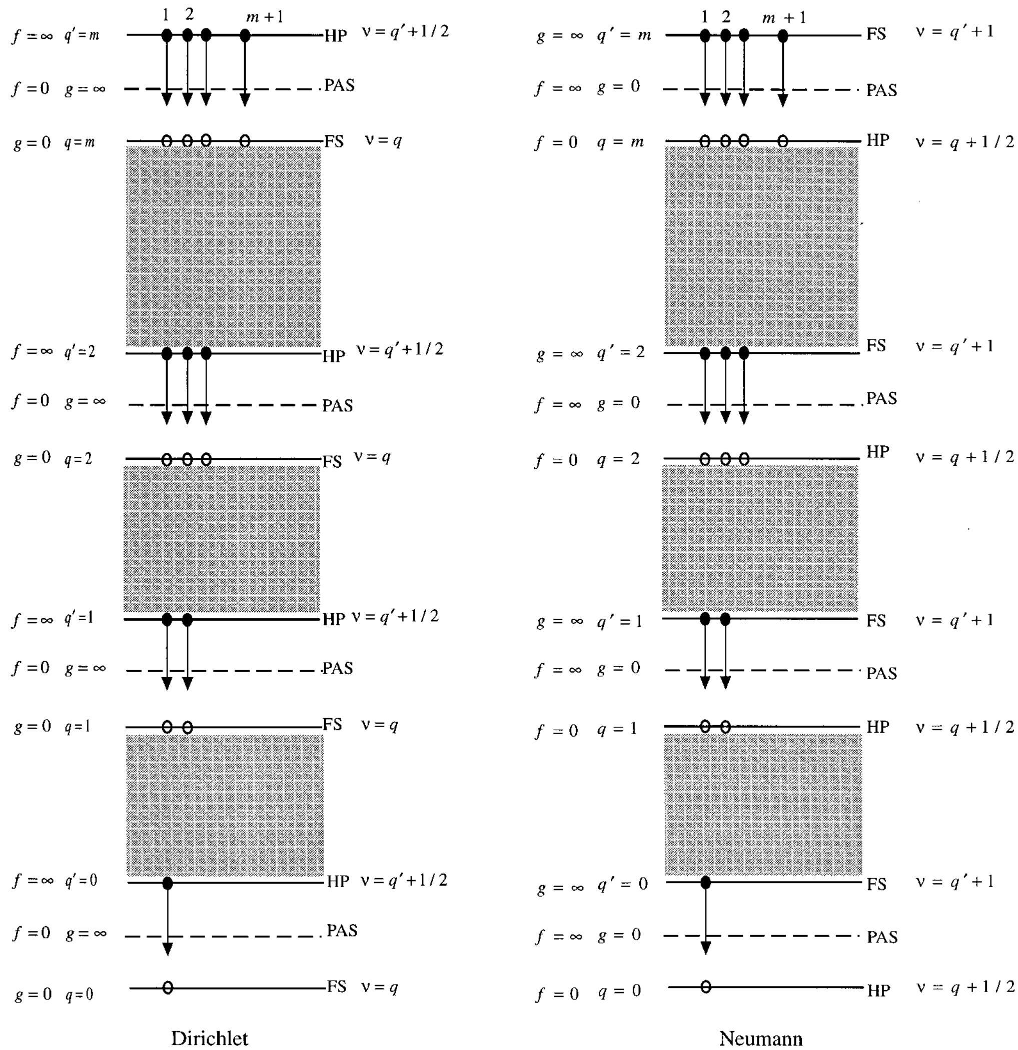

FIG. 4. This figure illustrates the eigenfunction flow discussed in the proof of Theorem 2.

$$\frac{\delta\alpha}{\delta\mu} = -\frac{\displaystyle\int_0^\pi \frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}\,\Theta^2(\vartheta)d\vartheta}{\displaystyle\int_0^\pi \frac{\kappa^2\sin^2(\vartheta)}{\sqrt{1-\kappa^2\cos^2\vartheta}}\,\Theta^2(\vartheta)d\vartheta} < 0,$$

thus $\delta\alpha/\delta g > 0$. As g increases from 0, which corresponds to the boundary condition on a PAS, to ∞, which corresponds to the boundary condition on a half-plane, the corresponding eigenvalues also increase. Here if the half-plane eigenvalue reached as $g \to \infty$ has the value $q' + 1/2$, we must have $\nu < q' + 1/2$. This completes the second part of the proof.

Finally, we must demonstrate that $q' = q$. Note that under the combined reversible flow illustrated by Fig. 4: g going from ∞ to 0 and f going from ∞ to 0, we convert a Dirichlet eigenfunction of the half-plane to one of the free-space problem, all the while reducing the eigenvalue from $q' + 1/2$ to q. Now the eigenfunction that starts at $q' = 0$ must become that of

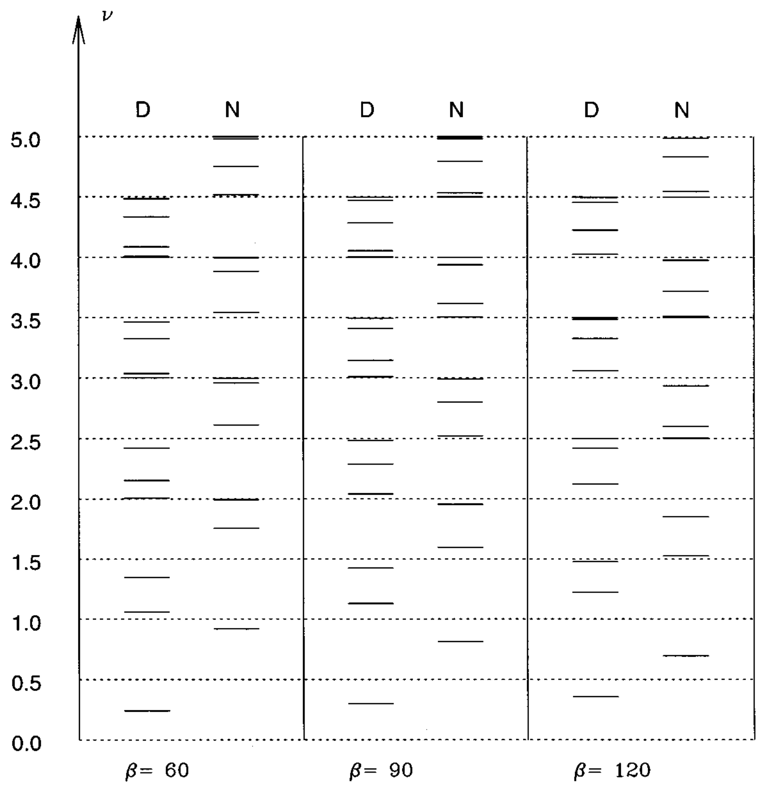

FIG. 5. This figure shows the location of the eigenvalues of Δ_Ω with boundary conditions on a plane angular sector with corner angles, 60°, 90°, and 120°. For Dirichlet boundary condition the eigenvalues, in columns marked ''*D*,'' lie between an integer and a half-integer, for Neumann boundary condition the eigenvalues, in columns marked ''*N*,'' lie between a half-integer and an integer.

$q=0$, since there are no other eigenvalues of the free-space problem below $\nu=1/2$. Next, the eigenfunctions at $q'=1$ (there are two such eigenvalues that are affected by the flow) must become those of $q=1$ (again there are two such) because there are no remaining eigenfunctions of the free space problem below $\nu=3/2$ once the $q=0$ mode has already been accounted for. In general then, the proof proceeds by induction. Assume we have accounted for all the modes up to $q'-1$. The eigenfunctions at q' (there are $q'+1$ such eigenfuntions that are affected by the flow) must flow to the q eigenfunctions (there are $q+1$ such eigenfunctions), since there are no other available modes with $\nu<q'+1/2$. Hence for all q', $q=q'$. For the PAS problem, this proves that $q\leqslant\nu\leqslant q+1/2$, completing the proof. Therefore, we conclude that the eigenvalues of Δ_Ω with a Dirichlet boundary condition specified on a PAS (segment of a great circle) lie between the free space eigenvalues (integers) and the eigenvalues of Δ_Ω when the Dirichlet boundary condition is specified on a half-plane (half-integers), see Fig. 5:

$$n<\nu<n+\tfrac{1}{2}.$$

□

The proof of this theorem for a Neumann boundary condition is very similar to that of a Dirichlet boundary condition and it is briefly outlined in the following. The free space boundary condition in the sphero–conal coordinate system appropriate for the Neumann case are given by

$$\Theta(0)=0,\quad \Theta(\pi)=0,\quad \Theta(0)=0,\quad \Phi(\pi)=0. \tag{21}$$

This boundary condition also corresponds to the odd Dirichlet boundary condition for a PAS. The Neumann boundary condition for a PAS is

$$\Theta(0)=0, \quad \Theta'(\pi)=0, \quad \Phi(0)=0, \quad \Phi(\pi)=0. \tag{22}$$

and the Neumann boundary condition for a half-plane is

$$\Theta(0)=0, \quad \Theta'(\pi)=0, \quad \Phi(0)=0, \quad \Phi'(\pi)=0. \tag{23}$$

If we define the following set of boundary conditions for the Φ equation:

$$\Theta(0)=0, \quad \Theta'(\pi)=0, \quad \Phi(0)=0, \quad \Phi'(\pi)=-f\,\frac{\Phi(\pi)}{\kappa},$$

then $f=0$ will correspond to the half-plane boundary condition and $f=\infty$ will correspond to the PAS boundary condition. The eigenvalues are the stationary value of Eq. (20). We have already proven that $\delta\alpha/\delta f$ is positive. Thus for a non-negative integer q if the half-plane eigenvalue reached as $f\rightarrow 0$ has the value $q+1/2$, we must have $\nu>q+1/2$. Next if we define the following set of boundary conditions for the Θ equation:

$$\Theta(0)=0, \quad \Theta'(\pi)=-g\,\frac{\Theta(\pi)}{\kappa'} \quad \Phi(0)=0, \quad \Phi(\pi)=0,$$

then $g=0$ corresponds to the PAS boundary condition and $g=\infty$ corresponds to the free space boundary condition. The eigenvalues are the stationary values of Eq. (18). In this case if the free space eigenvalue reached as $g\rightarrow\infty$ has the value $q'+1$, where q' is some other non-negative integer, then we must have $\nu<q'+1$. Again we must demonstrate that $q=q'$. Here also under the combined reversible flow: with f going from infinity to zero and g going from infinity to zero, we convert a Neumann eigenfunction for the free space problem to one of the half-plane problem, all the while reducing the eigenvalue from $q'+1$ to $q+1/2$, see Fig. 4. The eigenfunction that starts at $q'=0$ must become that of $q=0$, since there is no eigenvalue of the half-plane below $\nu=1$. Next, the eigenfunctions at $q'=1$ (there are two such eigenvalues that are affected by the flow) must become those of $q=1$ (again there are two such), because there are no remaining eigenfunctions of the half-plane problem below $\nu=2$ once the $q=0$ eigenfunction has already been accounted for. This process can be continued until all the modes up to $q'-1$ have been accounted for. At q', the q' eigenfunctions must flow to the q eigenfunctions, since there are no other available eigenfunctions of the half-plane problem with $\nu<q'+1$. Thus for all q', we must have $q'=q$. Therefore, we have proven that the eigenvalues of Δ_Ω with the Neumann boundary condition specified on a PAS (segment of a great circle) lie between the eigenvalues of Δ_Ω for a half-plane (half-integers) and the free space eigenvalues (integers), see Fig. 5:

$$n+\tfrac{1}{2}<\nu<n+1.$$

□

III. THE WKB EIGENVALUES

In order to study the solutions of Eqs. (3) and (4) for large ν, it is convenient to transform these equations to their Jacobian form.[8] This is accomplished in two steps. First, we set $\vartheta=\gamma-\pi/2$, and then use the transformation

$$\frac{d\gamma}{dt}=\sqrt{1-\kappa^2\sin^2\gamma},$$

which transforms Eq. (3) to

$$\frac{d^2\Theta}{dt^2}=(h-\nu(\nu+1)\kappa^2\ \mathrm{sn}^2(t))\Theta,$$

where $h=\nu(\nu+1)\kappa^2+\mu$ and sn is the Jacobian elliptic function. The above equation is of the general form

$$\frac{d^2w}{dx^2}=q(x)w=\{f(x)+g(x)\}w. \tag{24}$$

For small g/f, this type of equation has approximate solutions of the form

$$w_{1,2}(x)=f^{-1/4}(x)\exp\left\{\pm\int f^{1/2}(x)dx\right\},$$

in a given finite interval (a_1,a_2) provided that $f(\mathrm{x})$ is a real, twice continuously differentiable function, $g(x)$ a continuous real or complex function, and the error control function, $F(x)$, defined by

$$F(x)=\int\left\{\frac{1}{f^{1/4}}\frac{d^2}{dx^2}\left(\frac{1}{f^{1/4}}\right)-\frac{g}{f^{1/2}}\right\}dx,$$

in the absence of singularities, is bounded.[9] The boundedness of F guarantees that the approximate solution is asymptotically correct for large f. If the differential equation has a regular singularity, ie, $q(x)$ has a double pole, then $F(x)$ would be bounded only if $g(x)$ is chosen such that the coefficient of its singular part is precisely $-1/4$.[9] The Jacobian elliptic function $\mathrm{sn}^2(t)$ has a double pole which is relatively close to the real axis. Therefore, we write

$$h-\nu(\nu+1)\kappa^2\ \mathrm{sn}^2(t)=\kappa^2(\nu+\tfrac{1}{2})^2(1-\mathrm{sn}^2(t))+\mu-\frac{\kappa^2}{4}(1-\mathrm{sn}^2(t)),$$

and choose

$$f=\kappa^2(\nu+\tfrac{1}{2})^2(1-\mathrm{sn}^2(t))+\mu$$

and

$$g=-\tfrac{1}{4}(1-\mathrm{sn}^2(t))\kappa^2.$$

Note that the choice of the singular part of g is rather arbitrary as long as it does not grow with ν. Transforming back, we find the solution[9]

$$\Theta(\vartheta)=\frac{1}{\sqrt[4]{(\nu+\frac{1}{2})^2\kappa^2\ \sin^2\ \vartheta+\mu}}\cos\left\{\int_{\vartheta_0}^{\vartheta}\sqrt{\frac{(\nu+\frac{1}{2})^2\kappa^2\ \sin^2\ \vartheta+\mu}{1-\kappa^2\ \cos^2\ \vartheta}}\,d\vartheta+\delta_\vartheta\right\}, \tag{25}$$

and in a similar manner

$$\Phi(\varphi)=\frac{1}{\sqrt[4]{(\nu+\frac{1}{2})^2\kappa'^2\ \sin^2\ \varphi-\mu}}\cos\left\{\int_{\varphi_0}^{\varphi}\sqrt{\frac{(\nu+\frac{1}{2})^2\kappa'^2\ \sin^2\ \varphi-\mu}{1-\kappa'^2\ \cos^2\ \varphi}}\,d\varphi+\delta_\varphi\right\}. \tag{26}$$

For $\mu>0$, the turning point for the Θ equation, $\vartheta_0=0$, and the turning point for the Φ equation,

$$\varphi_0=\arcsin\sqrt{\frac{\mu}{\kappa'^2(\nu+\frac{1}{2})^2}}, \tag{27}$$

for $\mu<0$, $\varphi_0=0$ and

$$\vartheta_0=\arcsin\sqrt{\frac{-\mu}{\kappa^2(\nu+\frac{1}{2})^2}}. \tag{28}$$

For small g/f, g can be neglected in Eq. (24) which when transformed back gives Eq. (3) with $\nu(\nu+1)$ replaced by $(\nu+1/2)^2$. By using the transformation

$$v(\vartheta)=\sqrt[4]{1-\kappa^2\cos^2(\vartheta)}\Theta(\vartheta),$$

this latter equation can be converted to

$$\frac{d^2}{d\vartheta^2}v(\vartheta)+p(\vartheta)v(\vartheta)=0, \tag{29}$$

with

$$p(\vartheta)=\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{(1-\kappa^2\cos^2\vartheta)}. \tag{30}$$

By following the same procedure on the Φ equation we find

$$\frac{d^2}{d\varphi^2}u(\varphi)+p(\varphi)u(\varphi)=0, \tag{30}$$

where

$$u(\varphi)=\sqrt[4]{1-\kappa'^2\cos^2(\varphi)}\Phi(\varphi),\quad p(\varphi)=\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{(1-\kappa'^2\cos^2\varphi)}.$$

By using the Liouville transformation,[9]

$$y(x)=\left(\frac{dx}{d\vartheta}\right)^{1/2}v(\vartheta),$$

Eq. (29) can be transformed to

$$\frac{d^2y}{dx^2}=\left\{-\left(\frac{d\vartheta}{dx}\right)^2p(\vartheta)+\left(\frac{d\vartheta}{dx}\right)^{1/2}\frac{d^2}{dx^2}\left[\left(\frac{d\vartheta}{dx}\right)^{-1/2}\right]\right\}y.$$

The first term in the curly brackets can be set equal to any smooth function of x,[9] and for small μ/ν the second term can be ignored. We thus set

$$\left(\frac{d\vartheta}{dx}\right)^2p(\vartheta)=\left(\frac{x^2}{4}+a\right), \tag{31}$$

giving

$$\frac{d^2y}{dx^2}+\left(\frac{x^2}{4}+a\right)y=0,$$

which is the Weber equation. The parameter a is determined from Eq. (31) by requiring that the turning points of the Lamé equation and the Weber equation occur at the same time thus ensuring the regularity of $d\vartheta/dx$ at the turning points

$$\int_0^{\vartheta_0}\sqrt{\frac{\mu+(\nu+\frac{1}{2})^2\kappa^2\,\sin^2\,\vartheta}{1-\kappa^2\,\cos^2\,\vartheta}}d\vartheta=\int_0^{2i\sqrt{a}}\sqrt{a+\frac{x^2}{4}}\,dx. \tag{32}$$

In the above

$$\vartheta_0=\sin^{-1}\left(i\sqrt{\frac{\mu}{(\nu+1/2)^2\kappa^2}}\right).$$

Similarly, for the Φ equation we find

$$\frac{d^2z}{dx^2}+\left(\frac{x^2}{4}-a\right)z=0,$$

where now

$$z(x)=\left(\frac{dx}{d\varphi}\right)^{1/2}u(\varphi)\,.$$

The phase factors, δ_ϑ and δ_φ, are determined by matching the WKB solutions, Eqs. (25) and (26), to the asymptotic solutions of the Weber equation.[10] It is found that[11]

$$\delta_\varphi=\frac{\pi}{4}\mp\frac{3\pi}{8}+\frac{\phi_2(a)}{2}\mp\frac{1}{2}\,D(a)-\frac{a}{2}\,\ln|a|+\frac{a}{2},$$

$$\delta_\sigma=\frac{\pi}{4}\mp\frac{3\pi}{8}+\frac{\phi_2(-a)}{2}\pm\frac{1}{2}\,D(a)+\frac{a}{2}\,\ln|a|-\frac{a}{2},$$

where the upper signs are used for Dirichlet boundary condition, the lower signs are used for Neumann boundary condition and

$$D(a)=\arctan\left[\tanh\left(\frac{\pi a}{2}\right)\right],$$

and

$$\phi_2(a)=\arg\,\Gamma(\tfrac{1}{2}+ia).$$

The parameter a determined from Eq. (32) is given by

$$a=\frac{2}{\pi}\left\{\frac{(\nu+\frac{1}{2})^2\kappa'^2-\mu}{\kappa'\sqrt{(\nu+\frac{1}{2})^2\kappa^2+\mu}}\left\{\Pi\left(\frac{\pi}{2},\frac{\mu}{(\nu+\frac{1}{2})^2\kappa'^2},e\right)-K(e)\right\}\right\}, \tag{33}$$

where

$$e=\frac{1}{\kappa'}\sqrt{\frac{\mu}{(\nu+\frac{1}{2})^2\kappa^2+\mu}},\quad \mu>0,$$

and Π is the elliptic integral of the third kind. For $\mu<0$ a can be determined from the above equation by replacing μ by $|\mu|$ and κ' by κ. A similar relationship as Eq. (32), which also gives Eq. (33) for the parameter a, also holds for the Φ equation which guarantees the regularity of $d\varphi/dx$ at the turning points. We find the set of eigenvalue equations[11]

$$\begin{aligned} &J_\varphi+\phi_2(a)-D(a)-a\ \ln|a|+a=(m+\tfrac{1}{4})\pi,\quad m=0,1,\dots \\ &J_\vartheta-\phi_2(a)+a\ \ln|a|-a=(n+\tfrac{1}{2})\pi,\quad n=0,1,\dots \end{aligned}\tag{34}$$

for the Dirichlet boundary condition and

$$\begin{aligned} &J_\varphi+\phi_2(a)+D(a)-a\ \ln|a|+a=(m+\tfrac{3}{4})\pi,\quad m=0,1,\dots \\ &J_\vartheta-\phi_2(a)+a\ \ln|a|-a=(n+\tfrac{1}{2})\pi,\quad n=0,1,\dots \end{aligned}\tag{35}$$

for the Neumann boundary condition, where

$$J_\vartheta=\int_{\vartheta_0}^{\pi-\vartheta_0}\sqrt{\frac{(\nu+\frac{1}{2})^2\kappa^2\ \sin^2\ \vartheta+\mu}{1-\kappa^2\ \cos^2\ \vartheta}}\,d\vartheta,$$

and

$$J_\varphi=\int_{\varphi_0}^{\pi-\varphi_0}\sqrt{\frac{(\nu+\frac{1}{2})^2\kappa'^2\ \sin^2\ \varphi-\mu}{1-\kappa'^2\ \cos^2\ \varphi}}\,d\varphi,$$

can be expressed in terms of elliptic mtegrals,[12]

$$J_\vartheta=\frac{2\mu}{\kappa'\sqrt{(\nu+\frac{1}{2})^2\kappa^2+\mu}}\left\{\Pi\left(\frac{\pi}{2},\frac{(\nu+\frac{1}{2})^2\kappa^2}{(\nu+\frac{1}{2})^2\kappa^2+\mu},r_+\right)\right\},$$

and

$$J_\varphi=\frac{2\mu}{\kappa'\sqrt{(\nu+\frac{1}{2})^2\kappa^2+\mu}}\left\{\Pi\left(\frac{\pi}{2},\frac{(\nu+\frac{1}{2})^2\kappa'^2-\mu}{(\nu+\frac{1}{2})^2\kappa'^2},r_+\right)-K(r_+)\right\},$$

for $\mu>0$ and

$$J_\vartheta=\frac{-2\mu}{\kappa\sqrt{(\nu+\frac{1}{2})^2\kappa'^2-\mu}}\left\{\Pi\left(\frac{\pi}{2},\frac{(\nu+\frac{1}{2})^2\kappa^2+\mu}{(\nu+\frac{1}{2})^2\kappa^2},r_-\right)-K(r_-)\right\},$$

and

$$J_\varphi=\frac{-2\mu}{\kappa\sqrt{(\nu+\frac{1}{2})^2\kappa'^2-\mu}}\left\{\Pi\left(\frac{\pi}{2},\frac{(\nu+\frac{1}{2})^2\kappa'^2}{(\nu+\frac{1}{2})^2\kappa'^2-\mu},r_-\right)\right\},$$

for $\mu<0$. In the above

$$r_{+}=\frac{\kappa}{\kappa'}\sqrt{\frac{(\nu+\frac{1}{2})^2\kappa'^2-\mu}{(\nu+\frac{1}{2})^2\kappa^2+\mu}},$$

$$r_{-}=\frac{\kappa'}{\kappa}\sqrt{\frac{(\nu+\frac{1}{2})^2\kappa^2+\mu}{(\nu+\frac{1}{2})^2\kappa'^2-\mu}},$$

and K is the elliptic integral of the first kind. Using relations between elliptic integrals,[13] we find from the above equations

$$J_{\vartheta}+J_{\varphi}=(\nu+\tfrac{1}{2})\pi. \tag{36}$$

For the free space boundary condition, Eq. (21), we find the set of eigenvalue equations

$$\begin{aligned} J_{\varphi}+\phi_2(a)-D(a)-a\ \ln|a|+a&=(m+\tfrac{1}{4})\pi, \quad m=0,1,\dots\ , \\ J_{\vartheta}-\phi_2(a)+D(a)+a\ \ln|a|-a&=(n+\tfrac{1}{4})\pi, \quad n=0,1,\dots\ . \end{aligned} \tag{37}$$

Similarly, for both Dirichlet and Neumann boundary conditions, Eqs. (16) and (23), on a half-plane we find the set of eigenvalue equations

$$\begin{aligned} J_{\varphi}+\phi_2(a)-a\ \ln|a|+a&=(m+\tfrac{1}{2})\pi, \quad m=0,1,\dots\ , \\ J_{\vartheta}-\phi_2(a)+a\ \ln|a|-a&=(n+\tfrac{1}{2})\pi, \quad n=0,1,\dots\ . \end{aligned} \tag{38}$$

By adding the eigenvalue equations for the Dirichlet boundary condition, Eq. (34), and using Eq. (36), we obtain

$$\nu=m+n+\frac{1}{4}+\frac{D(a)}{\pi}, \tag{39}$$

and by adding the eigenvalue equations for the Neumann boundary condition, Eq. (35), we find

$$\nu=m+n+\frac{3}{4}-\frac{D(a)}{\pi}. \tag{40}$$

Similarly, for the free space boundary condition, Eq. (37), we find

$$\nu=m+n,$$

and for the half-plane boundary condition, Eq. (38), we find

$$\nu=m+n+\tfrac{1}{2}.$$

From the last two equations it can be seen that the WKB eigenvalue equations, Eq. (37), for free space boundary condition, and Eq. (38) for half-plane boundary condition, give the eigenvalues, ν, exactly: that is, the eigenvalues for the free space boundary condition calculated from Eq. (37) are exactly integers and the eigenvalues for half-plane boundary condition calculated from Eq. (38) are exactly half-integers. Furthermore, since

$$|D(a)|<\frac{\pi}{4},$$

TABLE I. Exact and WKB eigenvalues, Dirichlet boundary condition, $\beta=60°$.

Exact eigenvalues		WKB eigenvalues	
ν	μ	ν	μ
0.240 100	0.036 081	0.250 000	0.000 000
1.061 291	−0.738 682	1.056 152	−0.739 774
1.347 988	0.404 089	1.352 332	0.353 598
2.007 534	−2.798 184	2.005 562	−2.795 198
2.151 363	−0.448 877	2.152 149	−0.480 823
2.421 224	1.148 104	2.420 612	1.101 099
3.000 689	−6.411 688	3.000 415	−6.410 392
3.034 598	−2.299 160	3.031 866	−2.237 473
3.247 569	0.046 190	3.250 000	0.000 000
3.464 345	2.349 962	3.461 728	2.308 129
4.000 057	−11.542 236	4.000 029	−11.540 810
4.004 708	−5.536 257	4.003 667	−5.541 524
4.088 336	−1.575 254	4.087 239	−1.604 552
4.335 547	0.816 227	4.336 819	0.766 347
4.485 458	4.058 339	4.482 941	4.018 364

from Eqs. (39) and (40) we find for the Dirichlet boundary condition

$$m+n<\nu<m+n+\tfrac{1}{2},$$

and for the Neumann boundary condition

$$m+n+\tfrac{1}{2}<\nu<m+n+1,$$

in agreement with the results of theorem 2.

Limiting cases. Small a/ν. For small a/ν, J_ϑ and J_φ can be approximated by[11]

$$\begin{aligned} J_\vartheta&=2(\nu+\tfrac{1}{2})\left(\frac{\pi}{2}-\frac{\beta}{2}\right)+a\ \ln[8\kappa\kappa'(\nu+\tfrac{1}{2})]-a\ \ln|a|+a+O(\alpha^2),\\ J_\varphi&=2(\nu+\tfrac{1}{2})\left(\frac{\beta}{2}\right)-a\ \ln[8\kappa\kappa'(\nu+\tfrac{1}{2})]+a\ \ln|a|-a+O(\alpha^2), \end{aligned} \tag{41}$$

where

$$\alpha=\frac{2\kappa' a}{\kappa\sqrt{\nu(\nu+1)}}.$$

To this approximation the set of eigenvalue equations for the Dirichlet boundary condition on a plane angular sector, Eq. (34), becomes

$$2(\nu+\tfrac{1}{2})\left(\frac{\beta}{2}\right)-a\ \ln[8\kappa\kappa'(\nu+\tfrac{1}{2})]+\phi_2(a)-D(a)=(m+\tfrac{1}{4})\pi,\quad m=0,1,\dots\ ,$$

$$2(\nu+\tfrac{1}{2})\left(\frac{\pi}{2}-\frac{\beta}{2}\right)+a\ \ln[8\kappa\kappa'(\nu+\tfrac{1}{2})]-\phi_2(a)=(n+\tfrac{1}{2})\pi,\quad n=0,1,\dots\ .$$

By subtracting the above two equations we find

TABLE II. Exact and WKB eigenvalues, Dirichlet boundary condition, $\beta=90°$.

Exact eigenvalues		WKB eigenvalues	
ν	μ	ν	μ
0.296 584	0.089 456	0.299 781	0.080 958
1.131 248	−0.452 788	1.129 190	−0.438 011
1.426 512	0.917 647	1.427 775	0.899 504
2.039 575	−1.702 414	2.039 422	−1.684 193
2.287 571	0.216 125	2.287 856	0.213 095
2.480 880	2.667 648	2.487 856	2.648 919
3.009 062	−3.937 847	3.009 528	−3.919 378
3.146 403	−0.825 595	3.146 022	−0.818 730
3.408 679	1.533 190	3.408 992	1.523 359
3.495 891	5.437 690	3.495 505	5.417 830
4.001 846	−7.207 836	4.002 111	−7.187 871
4.053 806	−2.576 195	4.053 724	−2.565 033
4.284 205	0.332 446	4.284 291	0.330 690
4.470 929	3.789 494	4.470 831	3.777 082
4.499 185	9.222 705	4.499 019	9.201 634

$$2(\nu+\tfrac{1}{2})\left(\frac{\pi}{2}-\beta\right)+2a\ \ln[8\kappa\kappa'(\nu+\tfrac{1}{2})]-2\phi_2(a)+D(a)=(n-m+\tfrac{1}{4})\pi.$$

Substituting for $(\nu+\frac{1}{2})$ from Eq. (39), gives

$$2\left(m+n+\frac{3}{4}+\frac{D(a)}{\pi}\right)\left(\frac{\pi}{2}-\beta\right)+2a\ \ln\left[8\kappa\kappa'\left(m+n+\frac{3}{4}+\frac{D(a)}{\pi}\right)\right]-2\phi_2(a)+D(a)$$
$$=(n-m+\tfrac{1}{4})\pi. \tag{42}$$

In the same limit the set of eigenvalue equations for the Neumann boundary condition on a plane angular sector, Eq. (35), becomes

TABLE III. Exact and WKB eigenvalues, Dirichlet boundary condition, $\beta=120°$.

Exact eigenvalues		WKB eigenvalues	
ν	μ	ν	μ
0.356 355	0.158 119	0.358 126	0.174 722
1.226 096	−0.134 016	1.219 684	−0.090 849
1.476 873	1.536 303	1.480 990	1.538 258
2.123 472	−0.708 205	2.121 581	−0.659 142
2.417 310	1.067 192	2.419 379	1.087 689
2.497 681	4.406 617	2.498 460	4.406 798
3.057 603	−1.678 866	3.059 474	−1.634 478
3.327 410	0.488 369	3.326 217	0.525 600
3.486 757	3.648 295	3.488 577	3.657 995
3.499 779	8.787 899	3.499 890	8.787 049
4.023 832	−3.134 253	4.026 620	−3.093 377
4.227 428	−0.248 489	4.225 340	−0.202 646
4.456 604	2.761 217	4.458 635	2.781 612
4.498 416	7.777 962	4.498 879	7.784 635
4.499 983	14.671 188	4.499 992	14.669 535

TABLE IV. Exact and WKB eigenvalues, Dirichlet boundary condition, $\beta=60°$.

Exact eigenvalues		WKB eigenvalues	
ν	μ	ν	μ
0.919 039	−0.544 092	0.925 416	−0.562 905
1.756 877	0.031 801	1.750 000	0.000 000
1.992 189	−2.755 471	1.994 275	−2.762 736
2.612 874	0.907 904	2.611 660	0.862 343
2.960 147	−2.039 005	2.963 345	−2.059 364
2.999 308	−6.405 726	2.999 584	−6.406 754
3.542 794	2.211 193	3.545 553	2.160 504
3.880 762	−1.101 735	3.881 148	−1.131 761
3.995 194	−5.498 138	3.996 269	−5.510 924
3.999 942	−11.541 571	3.999 971	−11.540 469
4.515 746	3.988 468	4.518 469	3.936 896
4.752 602	0.035 213	4.750 000	0.000 000
4.980 945	−4.388 250	4.982 818	−4.410 636
4.999 484	−10.408 995	4.999 669	−10.415 666
4.999 995	−18.175 462	4.999 998	−18.173 528

$$2(\nu+\tfrac{1}{2})\left(\frac{\beta}{2}\right)-a\ \ln[8\kappa\kappa'(\nu+\tfrac{1}{2})]+\phi_2(a)+D(a)=(m+\tfrac{3}{4})\pi,\quad m=0,1,\dots\ ,$$

$$2(\nu+\tfrac{1}{2})\left(\frac{\pi}{2}-\frac{\beta}{2}\right)+a\ \ln[8\kappa\kappa'(\nu+\tfrac{1}{2})]-\phi_2(a)=(n+\tfrac{1}{2})\pi,\quad n=0,1,\dots\ .$$

By subtracting these two equations and using Eq. (40), we find

$$2\left(m+n+\frac{5}{4}-\frac{D(a)}{\pi}\right)\left(\frac{\pi}{2}-\beta\right)+2a\ \ln\left[8\kappa\kappa'\left(m+n+\frac{5}{4}-\frac{D(a)}{\pi}\right)\right]-2\phi_2(a)-D(a)$$
$$=(n-m-\tfrac{1}{4})\pi. \tag{43}$$

TABLE V. Exact and WKB eigenvalues, Dirichlet boundary condition, $\beta=90°$.

Exact eigenvalues		WKB eigenvalues	
ν	μ	ν	μ
0.814 655	−0.189 507	0.817 541	−0.183 573
1.597 131	0.795 774	1.595 459	0.782 870
1.955 326	−1.552 890	1.955 664	−1.535 754
2.520 877	2.621 752	2.521 225	2.602 671
2.801 149	−0.349 178	2.801 527	−0.346 344
2.990 672	−3.886 619	2.990 191	−3.865 543
3.504 197	5.424 529	3.504 590	5.403 516
3.617 052	1.261 783	3.616 648	1.254 866
3.938 056	−2.308 190	3.938 212	−2.297 999
3.998 143	−7.193 804	3.997 875	−7.171 821
4.500 819	9.219 426	4.500 985	9.197 701
4.532 032	3.680 927	4.532 113	3.668 644
4.795 768	−0.494 774	4.795 892	−0.492 960
4.984 161	−5.230 456	4.983 978	−5.216 220
4.999 642	−11.499 924	4.999 545	−11.467 714

TABLE VI. Exact and WKB eigenvalues, Dirichlet boundary condition, $\beta = 120°$.

Exact eigenvalues		WKB eigenvalues	
ν	μ	ν	μ
0.697 484	0.070 303	0.704 000	0.097 292
1.525 224	1.515 624	1.520 686	1.521 887
1.849 263	−0.498 577	1.853 450	−0.454 970
2.502 344	4.403 471	2.501 552	4.404 736
2.598 523	0.963 024	2.596 927	0.988 145
2.935 993	−1.528 545	2.934 267	−1.479 765
3.500 201	8.787 553	3.500 110	8.786 850
3.513 954	3.625 169	3.511 853	3.638 428
3.715 734	0.196 581	3.718 263	0.233 427
3.975 043	−2.930 602	3.972 115	−2.998 640
4.500 016	14.671 153	4.500 008	14.669 517
4.501 592	7.774 668	4.501 125	7.782 170
4.547 584	2.673 862	4.545 440	2.697 962
4.834 680	−0.881 895	4.836 155	−0.838 295
4.990 638	−4.876 280	4.988 412	−5.011 801

We note that Eqs. (42) and (43) are independent of ν. This allows us to solve these equations for a by performing a search in one dimension (as opposed to two dimensions when we need to find ν as well) and then use Eqs. (39) and (40) to determine ν.

Large a. As $a \to \infty$, $D(a) \to \pi/4 + O(e^{-\pi\alpha})$ and as $a \to -\infty$, $D(a) \to -\pi/4 + O(e^{-\pi|a|})$. By substituting these limiting values of $D(a)$ in the eigenvalue equations, Eqs. (34), (35), (37), and (38), we find that as $a \to \infty$ (μ positive), Eqs. (34) and (35) reduce to Eq. (38), and as $a \to -\infty$ (μ negative), Eqs. (34) and (35) reduce to Eq. (37). We already pointed out that the values of ν calculated from Eq. (38) are half-integers and those calculated from Eq. (37) are integers. This shows that for large positive values of a the eigenvalues, ν, approach half-integers and for large negative values of a they approach integers. This can be seen in Fig. 5 and Tables I–VI.

IV. NUMERICAL CALCULATION OF THE EIGENVALUES

The eigenvalues, (ν,μ), must be calculated by simultaneously requiring that the $\Phi(\varphi)$ solution is periodic with period 2π and the $\Theta(\vartheta)$ solution satisfies the boundary conditions. To do this, we used the following method: start with an initial ν, then use the shooting method[14] to find μ_ϑ for Eq. (3) and μ_φ from Eq. (4). Vary ν and find a new pair of $(\mu_\vartheta,\mu_\varphi)$. If the difference between μ_ϑ and μ_φ increases, vary ν in the opposite direction and find another pair of $(\mu_\vartheta,\mu_\varphi)$. Continue the process until the difference between μ_ϑ and μ_φ is small to a desirable limit. By using this method, we are able to find ν and μ accurate up to six decimal places. The WKB eigenvalues were calculated by applying the Newton–Raphson iteration to the set of equations given by Eqs. (34) and (35). Reference 14 has efficient routines for the calculation of elliptic integrals. These routines have been used in the WKB calculation of eigenvalues. For large values of ν and small values of a/ν, as was pointed out earlier, one does not need to solve the above set of the eigenvalue equations. Instead, for the Dirichlet boundary condition a can be determined from Eq. (42) and then Eq. (39) can be used to calculate ν; for the Neumann boundary condition a can be determmed from Eq. (43) and ν can be calculated from Eq. (40). It should be pointed out that in the Newton–Raphson iteration using Eqs. (34) and (35) the derivatives are respectively calculated from their approximate form for small a/ν [Eqs. (42) and (43)]. It is much easier to calculate derivatives of the latter equations, yet the convergence rate is equally good. Tables I–VI list the eigenvalues for Dirichlet and Neumann boundary conditions on a plane angular sector with corner angles of 60°, 90°, and 120°.

APPENDIX A: SPECIAL EIGENVALUES

Although, as pointed out in the text, the eigenvalues of Δ_Ω subject to the even Neumann and odd Dirichlet boundary conditions are of no mterest in this paper, for completeness we tabulate these eigenvalues for a 90° PAS along with the eigenvalues of Δ_Ω for a half-plane in Table VII. Note that the eigenvalues for the odd Dirichlet and even Neumann cases are all integers and those for the half-plane are half-integers. It should be pointed out that both odd Dirichlet and even Neumann boundary conditions on a half-plane result in the same set of eigenvalues.

TABLE VII. Special eigenvalues of Δ_Ω for a 90° PAS and a half-plane.

Even Neumann		Odd Dirichlet		Half-plane	
ν	μ	ν	μ	ν	μ
0.000 000	0.000 000	1.000 000	0.000 000	0.500 000	0.000 000
1.000 000	−0.500 000	2.000 000	−1.500 000	1.500 000	−0.866 025
1.000 000	0.500 000	2.000 000	1.500 000	1.500 000	0.866 025
2.000 000	−1.732 051	3.000 000	−3.872 984	2.500 000	−2.645 751
2.000 000	0.000 000	3.000 000	0.000 000	2.500 000	0.000 000
2.000 000	1.732 051	3.000 000	3.872 984	2.500 000	2.645 751
3.000 000	−3.949 490	4.000 000	−7.190 416	3.500 000	−5.431 181
3.000 000	−0.949 490	4.000 000	−2.190 416	3.500 000	−1.415 017
3.000 000	0.949 490	4.000 000	2.190 416	3.500 000	1.415 017
3.000 000	3.949 490	4.000 000	7.190 416	3.500 000	5.431 181
4.000 000	−7.211 103	5.000 000	−11.489 126	4.500 000	−9.221 070
4.000 000	−2.645 752	5.000 000	−5.196 152	4.500 000	−3.737 893
4.000 000	0.000 000	5.000 000	0.000 000	4.500 000	0.000 000
4.000 000	2.645 752	5.000 000	5.196 152	4.500 000	3.737 893
4.000 000	7.211 103	5.000 000	11.489 126	4.500 000	9.221 070

APPENDIX B: THE EXACT EIGENFUNCTIONS

1. The even Dirichlet case

a. The Θ equation

According to Refs. 8 and 4, the solution of Eq. (3) subject to the boundary conditions

$$\Theta(\pi)=0 \quad \text{Dirichlet boundary condition}$$

$$\Theta'(0)=0 \quad \text{even solution}$$

is given by the series

$$\Theta_e(\vartheta)=\sum_{m=0}^{\infty} A_m \cos(2m-\tfrac{1}{2})\vartheta.$$

Substituting the above series in Eq. (3) results in the recurrence relation

$$A_{m-1}\alpha_m+A_m\beta_m+A_{m+1}\gamma_m=0, \tag{B1}$$

where

$$\alpha_m=\frac{\kappa^2}{4}\left[\frac{(4m-3)(4m-5)}{4}-\nu(\nu+1)\right],$$

$$\beta_m = \left[\frac{(4m-1)^2}{4} \left(\frac{\kappa^2}{2} - 1 \right) + \frac{\nu(\nu+1)\kappa^2}{2} + \mu \right],$$

$$\gamma_m = \frac{\kappa^2}{4} \left[\frac{(4m+1)(4m+3)}{4} - \nu(\nu+1) \right].$$

The above is a three term recurrence relation which by a rather straightforward manipulation can be written in the form of a continued fraction

$$\frac{A_{m-1}}{A_m} = -\frac{\beta_m}{\alpha_m} + \cfrac{\frac{\gamma_m}{\alpha_m}}{\frac{\beta_{m+1}}{\alpha_{m+1}} + \cfrac{\frac{\gamma_{m+1}}{\alpha_{m+1}}}{\frac{\beta_{m+2}}{\alpha_{m+2}} + \cdots}}, \tag{B2}$$

or

$$\frac{A_{m+1}}{A_m} = -\frac{\beta_m}{\gamma_m} + \cfrac{\frac{\alpha_m}{\gamma_m}}{\frac{\beta_{m-1}}{\gamma_{m-1}} + \cfrac{\frac{\alpha_{m-1}}{\gamma_{m-1}}}{\frac{\beta_{m-2}}{\gamma_{m-2}} + \cdots}}. \tag{B3}$$

The above continued fractions converge rather fast, so approximately 20 terms are enough to achieve an accuracy of up to eight decimal places. Following Ref. 4, A_{11} is assumed to be unity and Eq. (B2) is used to calculate A_0. Then A_{-11} is assumed to be unity and Eq. (B3) is used to calculate A_0 again. The A_0 found by starting at A_{-11} is set equal to the A_0 calculated the first time and Eq. (B3) is used to scale A_{-1} through A_{-10}. Finally, all the coefficients are normalized to make

$$\Theta_e(0) = 1.$$

b. The Φ equation

The two independent solutions of Eq. (4) satisfying the even boundary conditions

$$\Phi'(0) = 0, \quad \Phi'(\pi) = 0$$

are given by

$$\Phi_{e1}(\varphi) = \sum_{m=0}^{\infty} B_{2m} \cos 2m\varphi,$$

and

$$\Phi_{e2}(\varphi) = \sum_{m=0}^{\infty} B_{2m+1} \cos(2m+1)\varphi.$$

The recurrence relations for the first solution are

$$B_0\left(\frac{\nu(\nu+1)\kappa'^2}{2}-\mu\right)+B_2\,\frac{\kappa'^2}{4}(2-\nu(\nu+1))=0\,,$$

$$B_0\left(\frac{-\nu(\nu+1)\kappa'^2}{2}\right)+B_2\left(4\left(\frac{\kappa'^2}{2}-1\right)+\frac{\nu(\nu+1)\kappa'^2}{2}-\mu\right)+B_4\,\frac{\kappa'^2}{4}(12-\nu(\nu+1))=0,$$

and

$$B_{2m-2}\alpha_{2m}+B_{2m}\beta_{2m}+B_{2m+2}\gamma_{2m}=0,\quad m\geqslant 2.$$

The recurrence relation for the second solution is

$$B_{2m-1}\alpha_{2m+1}+B_{2m+1}\beta_{2m+1}+B_{2m+3}\gamma_{2m+1}=0,\quad m\geqslant 1,$$

where

$$\alpha_{2m}=\frac{\kappa'^2}{4}[(2m-2)(2m-1)-\nu(\nu+1)],$$

$$\beta_{2m}=\left[(2m)^2\left(\frac{\kappa'^2}{2}-1\right)+\frac{\nu(\nu+1)\kappa'^2}{2}-\mu\right], \tag{B4}$$

$$\gamma_{2m}=\frac{\kappa'^2}{4}[(2m+2)(2m+1)-\nu(\nu+1)].$$

The continued fraction for the first solution is

$$\frac{B_{2m-2}}{B_{2m}}=-\frac{\beta_{2m}}{\alpha_{2m}}+\cfrac{\frac{\gamma_{2m}}{\alpha_{2m}}}{\frac{\beta_{2m+2}}{\alpha_{2m+2}}+\cfrac{\frac{\gamma_{2m+2}}{\alpha_{2m+2}}}{\frac{\beta_{2m+4}}{\alpha_{2m+4}}+\cdots}},\quad m\geqslant 2, \tag{B5}$$

and for the second solution is

$$\frac{B_{2m-1}}{B_{2m+1}}=-\frac{\beta_{2m+1}}{\alpha_{2m+1}}+\cfrac{\frac{\gamma_{2m+1}}{\alpha_{2m+1}}}{\frac{\beta_{2m+3}}{\alpha_{2m+3}}+\cfrac{\frac{\gamma_{2m+3}}{\alpha_{2m+3}}}{\frac{\beta_{2m+5}}{\alpha_{2m+5}}+\cdots}}. \tag{B6}$$

It was decided that it would be accurate enough to assume B_{42} to be unity and use Eq. (B5) along with the first two recurrence relations to find B_{40} through B_0. Similarly, B_{41} is assumed to be unity and Eq. (B6) is used to determine B_{39} through B_1. The coefficients are then normalized to make

$$\Phi_e(0)=1.$$

2. The odd Neumann case

a. The Θ equation

The solution of Eq. (3) subject to the boundary conditions

$$\Theta'(\pi)=0 \quad \text{Neumann Boundary condition}$$

$$\Theta(0)=0 \quad \text{odd solution}$$

is

$$\Theta_o(\vartheta)=\sum_{m=0}^{\infty} A_m \sin(2m-\tfrac{1}{2})\vartheta.$$

The recurrence relation and the expression for the continued fraction for this equation are the same as for the Dirichlet case [Eqs. (B1), (B2), (B3)]. The coefficients are determined in the same manner, except that now they are normalized to make

$$\Theta_o'(0)=1.$$

b. The Φ equation

The two independent solutions of Eq. (4) satisfying the odd boundary conditions

$$\Phi(0)=0, \quad \Phi(\pi)=0$$

are given by

$$\Phi_{o1}(\varphi)=\sum_{m=0}^{\infty} B_{2m} \sin 2m\varphi,$$

and

$$\Phi_{o2}(\varphi)=\sum_{m=0}^{\infty} B_{2m+1} \sin(2m+1)\varphi .$$

The recurrence relation for the first solution is

$$B_{2m-2}\alpha_{2m}+B_{2m}\beta_{2m}+B_{2m+2}\gamma_{2m}=0, \quad B_0=0$$

and for the second solution is

$$B_1\left(\left(\frac{\kappa'^2}{2}-1\right)+\frac{3\nu(\nu+1)\kappa'^2}{4}-\mu\right)+B_3\frac{\kappa'^2}{4}(6-\nu(\nu+1))=0,$$

$$B_{2m-1}\alpha_{2m+1}+B_{2m+1}\beta_{2m+1}+B_{2m+3}\gamma_{2m+1}=0, \quad m\geqslant 1,$$

where α_{2m}, β_{2m} and γ_{2m} are given by Eq. (B4) and the continued fractions are given by Eqs. (B5), and (B6). The coefficients are determined in the same way as the coefficients for the Dirichlet case and then they are normalized to make

$$\Phi_o'(0)=1 .$$

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