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Noise and Nonlinearities in (1 + 1)-Dimensional System

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Abstract

We show that the Fokker-Planck equation can be applied to noisy sine-Gordon systems, either through reduction to soliton parameter or by the techniques of functional analysis. We also suggest approaches to introducing noise in the auxiliary Faddeev-Takhtadzhyan scattering equations.

I. Introduction

As the size of a physical system decreases the dependence of its transport characteristics on nonlocal and nonlinear phenomena becomes more pronounced. For example, although the electrical characteristics of a transistor may be determined by microscopic structure within a few measurements of bulk properties, electron correlations in the plane of the interface amplify fluctuations. Furthermore, self-field effects can produce strong nonlinearities. One example of these is exciton transport in molecular crystals. Both the field-like nature of the fluctuations and the nonlinear interaction between electromagnetic fields and charge transport may pose difficult problems in description and analysis of the dynamics of the system.

II. Relations Between the Physics of the Long Josephson Junction and the Mathematics of the sine-Gordon Equation

The effects of perturbations on the dynamics of nonlinear extended systems may be understood in the context of some actual structures in which the electrical behavior is accurately given by a simple description. One such system is the Josephson transmission line in which the dynamics are described by a perturbed sine-Gordon equation[1,2]. Although the sine-Gordon equation is deceptively simple the mathematical structure of its solution space is very rich[2–7].

The electron phase difference across a one-dimensional Josephson junction may be described by a sine-Gordon equation in the phase, $\phi(x,t)$, with additional nonzero terms for dissipation and driving current[1]:

$$\phi_{xx} - \phi_{tt} - \sin(\phi) = \alpha\phi_t - \gamma \,. \qquad (1)$$

In this equation the driving term γ may be written as the sum of applied current $\eta(x, t)$ and noise $\xi(x, t)$. Because a real junction is of finite length L, boundary conditions must be specified. There are generally three types:

1). Periodic $e^{i\phi(x)} = e^{i\phi(x+L)}$;

2). Reflective $\phi_x(0) = \phi_x(L) = C$;

3). Free $\phi_x(0) = \phi_{xx}(0)$, $\phi_x(L) = \phi_{xx}(L)$.

Condition 1) arises in an annular junction system [5], condition 2) arises when magnetic flux is assumed to be confined to the junction[8], and condition 3) arises from a lumped circuit equivalent of the device[9]. Generally, because a field radiates from the end of an oscillating junction the actual boundary condition is intermediate between 2) and 3)[1].

The solutions of the sine-Gordon equation (right-hand side of Eq. (1) set to zero) on the infinite line may be written in terms of solitary waves or solitons. These soliton solutions carry over to the perturbed sine-Gordon equation [Eq. (1)] on a finite interval [conditions 2) and 3)] if the perturbation is small and the interval is long enough. Furthermore, the soliton solutions of the sine-Gordon equation correspond, in a long Josephson junction to discrete packets of magnetic flux (fluxons) which propagating along the junction. Because dissipation terms in a long Josephson junction are generally small in magnitude, many of the most interesting properties of freely propagating fluxons far from boundaries are derived from the properties of soliton solutions of the pure sine-Gordon equation. These packets, induced by minute current vortices, behave like solitons and exhibit the same particle-like properties[1,8]. The literature associated with the pure sine-Gordon equation is quite extensive and generally treats the case of the sine-Gordon equation with either periodic boundary conditions or distribution-like conditions on the infinite line, ($\phi_t, \phi \to 0$ as $x \to \pm\infty$). Solutions of the pure sine-Gordon equation may be found by separation of space and time variables[3]. Generalizations of this technique yield multisoliton solutions which may be expressed in terms of the Jacobi theta functions and the Riemann-theta functions[4-6]. Solutions for the sine-Gordon equation generalized to two or more spatial dimensions are also given in terms of the Riemann theta function[4]. The main approach to derivation of multisoliton is through inverse scattering theory[5-7].

Numerical modeling experiments demonstrate the solitary wave properties of fluxon solutions of Eq. (1). Furthermore, numerical techniques show that driving currents and boundary conditions may exhibit quite complex interactions; in fact boundary conditions may constitute a much stronger perturbation on the solutions of the pure sine-Gordon system than do dissipation or driving currents[9].

III. Effects of Noise

We can assume that a system described by a finite number of variables, $q_1, \cdots, q_n$, dynamics as given by the Langevin equations

$$q_\nu = f_\nu(q_1, \cdots, q_n) + g_\nu^i \xi_i(t) ,$$

where the first term on the right-hand side gives the unperturbed dynamics.

specifies the effects of the Gaussian noise processes

$$\xi_i(t),\ \langle \xi_i(t) \rangle = 0$$

and

$$\langle \xi_i(t)\xi_k(0) \rangle = \delta_{ik}\delta(t) .$$

This equation may be applied to a spatially distributed system by reduction to a finite dimensional system. One particular technique is the finite element method. In our case we are interested in the finite dimensional description of the solutions of the dynamical equations given by phase and velocity of the solitons.

If the f_ν in Eq. (2) is sufficiently differentiable and there is a probability distribution P on the phase space satisfying Chapman-Kolomogorov conditions then P satisfies the Fokker-Planck equation[10,11]

$$\frac{\partial P(q,\, t)}{\partial t} = \left\{ -\frac{\partial}{\partial q_\nu} K_\nu(q) - \frac{1}{2}\,\frac{\partial^2}{\partial q_\nu \partial q_\mu} Q_{\nu\mu}(q) \right\} , \tag{3}$$

where

$$K_\nu(q) = f_\nu(q) + \frac{1}{2}\frac{\partial g^i_\nu}{\partial q_\mu} g^k_\mu(q)\delta_{ik} ,$$

$$Q_{\nu\mu} = g^i_\nu g^k_\mu \delta_{ik} .$$

We may apply this equation to perturbation of the motion of a single fluxon in the Josephson transmission line because the perturbation theory reduces Eq. (1) to a system of two ordinary differential equations. According to McLaughlin and Scott[8]

$$\dot{u} = \frac{\pi}{4}(\eta + \xi)(1 - u^2)^{3/2} - \alpha u(1 - u^2) , \qquad \dot{X} = u , \tag{4}$$

where u is the fluxon velocity, X is the phase, η and ξ are as above, $\eta = \cos t$.

We study the momentum q, which is defined by

$$q = \frac{8u}{(1 - u^2)^{1/2}} . \tag{5}$$

Making use of Eq. (2) and adding noise $\xi(t)$ obtain the Langevin equation

$$\begin{aligned} &\frac{dq}{dt} = f(q) + g(q)\xi(t) , \\ &f(q) = 2\pi\eta - \alpha q , \qquad g(q) = 2\pi . \end{aligned} \tag{6}$$

The associated Fokker–Planck equation is

$$\frac{\partial P(q,\, m,\, t)}{\partial t} = \left[-\frac{d}{dq}K(q) - \frac{m}{2}Q\frac{d^2}{dq^2} \right] P(q,\, m,\, t) ,$$

where $K = f(q)$, $Q = 1$, and m is a parameter indicating the intensity of the noise $\xi(t$ associated Hamiltonian is

$$H(p,\, q) = K(q)\,p + \frac{1}{2}Qp^2 .$$

We obtain the following results: 1). the Hamiltonian system associated to Eq. (8) is completely integrable; 2). in the time-independent steady state, the weak noise limit of solution to Eq. (7) is of the following form:

$$P(q,\, m) = N(m)Z(q)\exp\left[-\frac{L}{m}\right] , \tag{9}$$

where

$$L = \frac{(-2\pi\eta + \alpha q)^2}{\alpha} ,$$

$$Z(q) = \left(\frac{\alpha}{\pi m}\right)^{1/2}$$

and $N(m) = 1$; and 3). the time-dependent probability distribution of soliton velocity is given by the formula

$$P(u,\, m,\, t) = \frac{(1-u^2)^{3/2}}{8} \times \left[\phi_0^2(R) + \sum_{J=1}^{\infty} \phi_0(R)(-1)^j \frac{(2j)!}{j!} H_{2j}\left(\sqrt{\frac{\alpha}{m}}\right) \exp\left(-\frac{\alpha}{2m}R^2 - \frac{2j}{\alpha}t\right)\right] \tag{10}$$

with

$$R = \frac{8u}{(1-u^2)^{1/2}} - \frac{8u_\infty}{(1-u_\infty^2)^{1/2}} ,$$

ϕ_0 being the time-independent solution, u_∞ the stationary velocity and H_j the j-th Hermitian polynomial.

It is shown in Refs. [13] and [14] that equation (7) is the precise solution in the time-independent steady state and that the power series expansion (10) can be analytically calculated. With respect to expansion, five kinds of points in variable space, i.e., hyperbolic stable and unstable points, saddles, bifurcation singular points, regular points, and singular points of higher orders, are distinguished.

The exact (functional) form of the Fokker–Planck equation is also available to treat Eq. (1) directly. Assuming for a given noise process ξ, a probability distribution $P_\xi(\phi,\, \phi_t,\, t)$ on the space of solutions for Eq. (1), we can write a conservation equation for the functional P_ξ as follows:

$$\int \frac{\partial P_\xi}{\partial t} + \left[\frac{\delta}{\delta\phi}(\phi_t P_\xi) + \frac{\delta}{\delta\phi_t}(\phi_{tt} P_\xi)\right] dx = 0 . \tag{11}$$

Proceeding formally, as in Graham[11]

$$P_\xi(\phi,\, \phi_t,\, t) = T\left[\exp\int^{t_0} d\tau(\theta_1 + \theta_2)\right] P_\xi(\varphi,\, \dot\varphi_t,\, 0) , \tag{12}$$

where θ_1 and θ_2 are the functional operators

$$\int \frac{\delta}{\delta\phi}\phi dx \quad \text{and} \quad \int \frac{\delta}{\delta\phi_t}\varphi_{tt} dx$$

respectively. Expanding, and performing the ensemble average over noise processe obtain the functional equation:

$$\frac{\partial P}{\partial t} = -\left\{\int dx \frac{\delta}{\delta\phi}K_1 + \int dx \frac{\delta}{\delta\phi_t}\right\}P + \frac{1}{2}\left\{\iint dx_1 dx_2 \frac{\delta}{\delta\phi_\mu \delta\phi_\nu} Q_{\mu\nu}\right\}P$$

IV. Further Generalization

Faddeev and his coworkers[7] have derived the inverse scattering equations associated with the pure sine-Gordon equation, and used them to obtain explicit solutions in terms of action-angle variables. The most general form of the associated linear equations is as follows:

$$\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{d}{dt}+\frac{i}{4}w\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+\frac{1}{16\lambda}\begin{pmatrix} e^{iu} & 0 \\ 0 & e^{-iu} \end{pmatrix}-\lambda\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}=0\,, \quad (14)$$

$$\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{d}{dx}+\frac{i}{4}w\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}-\frac{1}{16\lambda}\begin{pmatrix} e^{iu} & 0 \\ 0 & e^{-iu} \end{pmatrix}-\lambda\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}=0\,, \quad (15)$$

where λ is the eigenvalue and (ψ_1, ψ_2) is the eigenfunction[6-8]. The consistency of these two linear equations implies that

$$u_t + u_x = w\,, \qquad w_t - w_x = \sin u\,, \quad (16)$$

i.e., u satisfies the pure sine-Gordon equation. Forest and his coworkers[5,6] have used these equations to study the perturbed sine-Gordon equation with periodic boundary conditions. If we rewrite the linear equations

$$\begin{aligned} \frac{d\psi_1}{dt}+\frac{i}{4}w\psi_1+\left(\frac{1}{16\lambda}e^{-iu}+\lambda\right)\psi_2=0\,, \\ -\frac{d\psi_2}{dt}+\frac{i}{4}w\psi_2+\left(\frac{1}{16\lambda}e^{iu}+\lambda\right)\psi_1=0\,, \end{aligned} \quad (17)$$

etc. Then we see that the linear equations appear as generalized two-level equations. This suggests an approach to noise and fluctuations via inverse scattering theory.

If we make the ansatz

$$\psi_i = \rho_i^{1/2} e^{-i\phi_i}\,,$$

where ρ_i is regarded as a number density and ϕ_i as a phase then we obtain eight coupled equations in number density and phase from Eq. (17). Writing the time equations for phase explicitly

$$\frac{d\phi_1}{dt}-\frac{w}{4}+K_{1,R}\left(\frac{\rho_2}{\rho_1}\right)\sin(\phi_2-\phi_1)-K_{1,C}\left(\frac{\rho_2}{\rho_1}\right)\cos(\phi_2-\phi_1)=0\,, \quad (18)$$

we see that fluctuations may be introduced into the system via fluctuations in number densities. This gives a means to measure departures from pure sine-Gordon dynamics within a functional framework consistent with inverse scattering theory. The essential step is to note that both the Faddeev-Takhtadzhyan inverse scattering equations and Landau-Ginzberg equations for an extended Josephson junction arise from functional equations[12].

$$ih\frac{\partial}{\partial t}\psi_i = \frac{\delta}{\delta\psi_i}F_T(\psi_1, \psi_2; A)\,, \quad (19)$$

where F_T is an expression for free energy and A is the electromagnetic field inside the junction.

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