The Use of a New Formula in Calculation of Scattering from Sinusoidal Surfaces

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Introduction

The problem of scattering of waves from periodic surfaces, especially sinusoidal surfaces has attracted a fair amount of interest. This problem was first treated by Lord Rayleigh [1] who used the periodic nature of the surface and the boundary conditions on the surface to reduce the problem to a set of algebraic equations for the so-called reflection coefficients. He then proceeded to find these coefficients by an approximate method. Throughout the years many authors have treated this problem by using approximate methods like small perturbations when the surface height is small, and physical optics "Kirchhoff's method" when the wavenumber is large compared with the curvature of the surface. More recently, Holford [2] has been able to solve this problem exactly by an integral equation method for both Dirichlet and Neumann boundary conditions. Although this problem has been solved for the acoustic waves, the two-dimensional nature of the boundary makes it possible to solve the same problem for the electromagnetic waves: that is, solve the scalar wave equation for the tangential component of the E-field subject to Dirichlet boundary condition, and for the tangential component of the H-field subject to Neumann boundary condition.

An approximate formula for scattering amplitude was developed by Dashen and Wurmser [3] which agrees with the Kirchhoff's solution (physical optics) for small surface curvature and with the small perturbation solution for small surface heights. We applied this formula to a sinusoidal surface and computed the normalized scattering amplitude squared as a function of the ratio of surface amplitude to surface wavelength for a range of frequencies. We repeated the same computation using the Kirchhoff's formula, perturbation theory, and small slope approximation [4]. Comparison of the above results with Holford's exact results shows that the solution obtained by the Dashen-Wurmser method is more accurate than the solution obtained from the other three approximate methods in all regimes. Furthermore, unlike

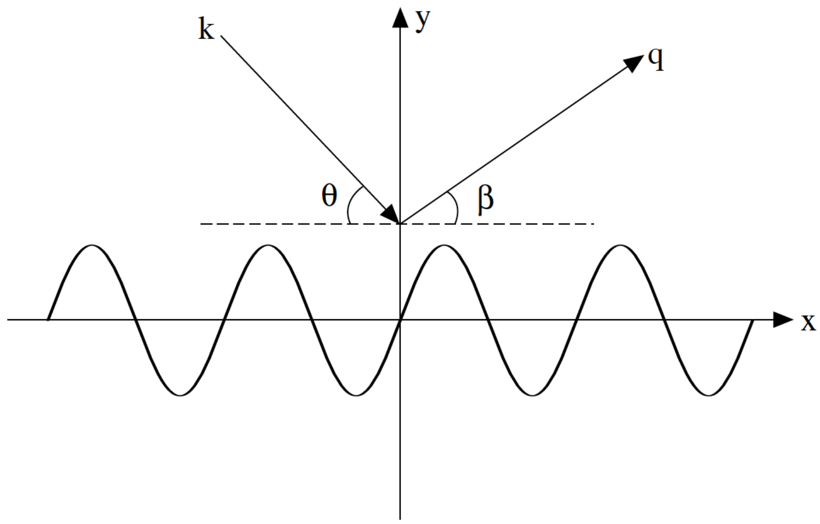

Figure 1: *The geometry of the problem.*

the Kirchhoff's solution, this solution approaches the perturbation theory solution for small surface heights. The Dashen-Wurmser formulation like those of Kirchhoff's does not account for non-local effects like shadowing and multiple scattering. As the surface height increase, these effects begin to emerge, at which point the Dashen-Wurmser solution begins to deviate from the exact solution. The derivation and motivation for Dashen and Wurmser formula are given in [5], [3], [6].

Theory

Let the sinusoidal surface S be parameterized by the vector $\mathbf{x} = x\hat{\mathbf{x}} + \xi(x)\hat{\mathbf{y}}$, (the surface does not vary along the z-axis). The scattering, therefore, is considered in the x-y plane which is also the plane of incidence. The unit normal $\hat{\mathbf{n}} = (-d\xi/dx\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{1 + (d\xi/dx)^2}$ points into the scattering region. The incident and outgoing waves are described by the the two dimensional wavevectors $\mathbf{k}$ and $\mathbf{q}$, respectively. These vectors have the same magnitude $|\mathbf{k}| = |\mathbf{q}| = k$, $k = 2\pi/\lambda$, where λ is the radiation wavelength. A bold variable is a two dimensional vector. For this calculation we have used a sinusoidal surface described by $\xi(x) = H\sin(px) = H\sin(2\pi x/\Lambda)$. The incident angle, θ, is measured from the negative x-axis in the clockwise direction and the scattering angle, β, is measured from the positive x-axis in the counter clockwise direction as illustrated by Fig(1).

We therefore have:

$$\mathbf{k} = k(\cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{y}}) = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}},$$
$$\mathbf{q} = k(\cos\beta\hat{\mathbf{x}} + \sin\beta\hat{\mathbf{y}}) = q_x\hat{\mathbf{x}} + q_y\hat{\mathbf{y}}.$$

Let us define the vectors **Q** and **W** such that:

$$\begin{aligned}\mathbf{Q} &= \mathbf{k}-\mathbf{q} = Q_x\hat{\mathbf{x}} + Q_y\hat{\mathbf{y}},\\ \mathbf{W} &= \mathbf{k}+\mathbf{q} = W_x\hat{\mathbf{x}} + W_y\hat{\mathbf{y}}.\end{aligned}$$

Where,

$$\begin{aligned}Q_x &= k(\cos\theta - \cos\beta),\\ Q_y &= -k(\sin\theta + \sin\beta),\\ W_x &= k(\cos\theta + \cos\beta),\\ W_y &= k(\sin\beta - \sin\theta).\end{aligned}$$

Then according to the new formula the scattering amplitude is given by [3]:

$$T = -i\int_{surface} G(\mathbf{n},\mathbf{k},\mathbf{q})e^{i\mathbf{Q}\cdot\mathbf{X}}dS, \tag{1}$$

where

$$G(\mathbf{n},\mathbf{k},\mathbf{q}) = \pm Q_n + \frac{W_n^2 - Q_n^2 + Q^2}{\sqrt{Q^2 - Q_n^2}}\tan^{-1}\left[\frac{\sqrt{Q^2-Q_n^2}}{Q_n}\right],$$

and

$$\begin{aligned}Q_n &= \mathbf{Q}\cdot\hat{\mathbf{n}},\\ W_n &= \mathbf{W}\cdot\hat{\mathbf{n}}.\end{aligned}$$

The plus sign in the above equation is used for Dirichlet (soft), and the minus sign is used for Neumann (hard) boundary condition. The normalization of T is such that $|iT/4|^2$ gives the differential scattering cross section. We have

$$dS = \sqrt{1 + (d\xi/dx)^2}dx$$

then (1) for an infinitely long surface becomes:

$$T = -i\int_{-\infty}^{\infty} G(\mathbf{n},\mathbf{k},\mathbf{q})e^{i(Q_x x + Q_y\xi(x))}\sqrt{1+\left(\frac{d\xi}{dx}\right)^2}dx. \tag{2}$$

The above integral can be broken up into an infinite sum of integrals each over one period of the surface:

$$T = -i\sum_{n=-\infty}^{\infty}\int_{\frac{2n\pi}{p}}^{\frac{2(n+1)\pi}{p}} G(\mathbf{n},\mathbf{k},\mathbf{q})e^{i(Q_x x + Q_y\xi(x))}\sqrt{1+\left(\frac{d\xi}{dx}\right)^2}dx. \tag{3}$$

If $x \to x - \dfrac{2n\pi}{p}$, since the surface is periodic, $\xi(x - \dfrac{2n\pi}{p}) = \xi(x)$, then (3) becomes

$$\begin{aligned} T &= -i \sum_{n=-\infty}^{\infty} e^{-iQ_x \frac{2n\pi}{p}} \int_0^{\frac{2\pi}{p}} G(\mathbf{n}, \mathbf{k}, \mathbf{q}) e^{i(Q_x x + Q_y \xi(x))} \sqrt{1 + \left(\frac{d\xi}{dx}\right)^2} dx \\ &= -i \left[\int_0^{\frac{2\pi}{p}} G(\mathbf{n}, \mathbf{k}, \mathbf{q}) e^{i(Q_x x + Q_y \xi(x))} \sqrt{1 + \left(\frac{d\xi}{dx}\right)^2} dx \right] \sum_{n=-\infty}^{\infty} e^{-iQ_x \frac{2n\pi}{p}} \\ &= -i \left[\int_0^{\frac{2\pi}{p}} G(\mathbf{n}, \mathbf{k}, \mathbf{q}) e^{i(Q_x x + Q_y \xi(x))} \sqrt{1 + \left(\frac{d\xi}{dx}\right)^2} dx \right] \delta(\frac{Q_x}{p} + m). \end{aligned} \tag{4}$$

In the above equation T will be zero unless $Q_x = -mp$, where m is an integer. This implies that Q_x can only take discrete values determined by the equation

$$Q_x = Q_{x_m} = k(\cos\theta - \cos\beta_m) = -mp = -\frac{2m\pi}{\Lambda},$$

for all other values of n the above sum is exactly zero. Since $k = 2\pi/\lambda$, the above equation becomes

$$\cos\beta_m = \cos\theta + m\frac{\lambda}{\Lambda}, \tag{5}$$

which is the familiar grating equation in optics. Similarly,

$$Q_y = Q_{y_m} = k(\sin\theta + \sin\beta_m).$$

From (4) we conclude that the incident wave is scattered in discrete directions determined by (5) as shown in Fig.(2).

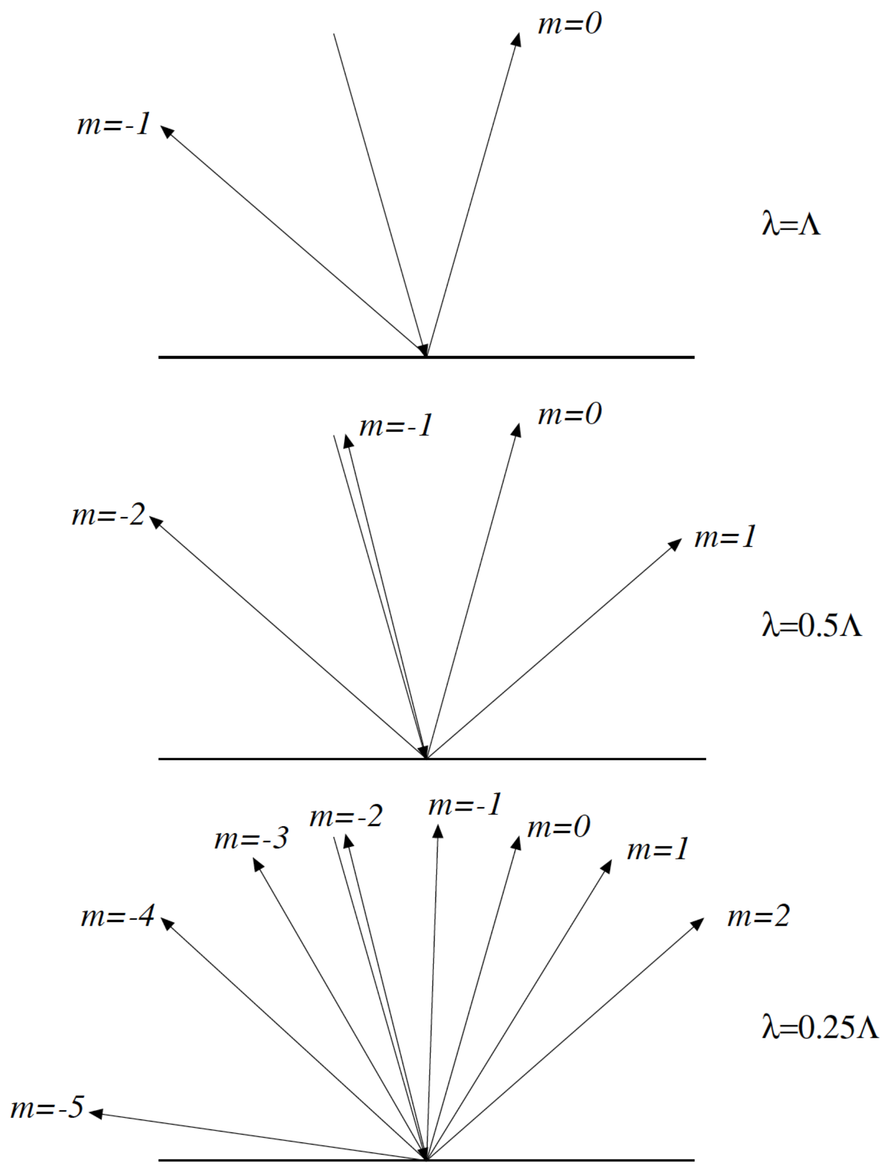

Figure 2: *Reflection of an incident wave from an infinite sinusoidal surface occurs in discrete directions. The number and angles of reflection is determined by the angle of incidence and the ratio of the radiation wavelength,* λ*, to the surface wavelength,* Λ*, according to Equation (5). The incident angle in this case is 75 degrees.*

The scattering amplitude for the scattered wave in each direction is therefore given by

$$T_m = \int_0^{\frac{2\pi}{p}} G(\mathbf{n}, \mathbf{k}, \mathbf{q}) e^{i(Q_{x_m}x + Q_{y_m}\xi(x))} \sqrt{1 + \left(\frac{d\xi}{dx}\right)^2} dx. \tag{6}$$

Results

For a given angle of incidence and the ratio λ/Λ we have calculated T_m by integrating (6) numerically. We have also calculated T_m using the Kirchhoff formula:

$$T_m^{Kirch.} = Q_n \int_0^{\frac{2\pi}{p}} e^{i(Q_{x_m}x + Q_{y_m}\xi(x))} \sqrt{1 + \left(\frac{d\xi}{dx}\right)^2} dx, \tag{7}$$

and first order perturbation theory :

$$T_m^{Pert.} = -4k_y q_y \int_0^{\frac{2\pi}{p}} \xi(x) e^{iQ_x x} dx \tag{8}$$

for Dirichlet boundary condition and

$$T_m^{Pert} = -4\left(k^2 - k_x q_x\right) \int_0^{\frac{2\pi}{p}} \xi(x) e^{iQ_x x} dx \tag{9}$$

for Neumann boundary condition [5],[3],[6]. Similarly, in this notation the small slope approximation [4] for Dirichlet boundary condition is given by

$$T_m^{ss} = \frac{4ik_y q_y}{Q_y} \int_0^{\frac{2\pi}{p}} e^{i(Q_{x_m}x + Q_{y_m}\xi(x))} dx. \tag{10}$$

and for Neumann boundary condition is given by

$$T_m^{ss} = \frac{4i\left(k^2 - k_x q_x\right)}{Q_y} \int_0^{\frac{2\pi}{p}} e^{i(Q_{x_m}x + Q_{y_m}\xi(x))} dx. \tag{11}$$

These results are compared with Holford's exact results [7]. Holford has computed $|S_m|^2$ which is related to $|T_m|^2$ by

$$|S_m|^2 = \frac{\gamma_0}{\gamma_m} |T_m|^2,$$

where

$$\gamma_m = \sin \beta_m.$$

In the following pages we compare $|S_m|^2$ for indicated values of m as a function surface amplitude to surface wavelength H/Λ, using the exact method, our method, (6), Kirchhoff's method, (7), perturbation theory, (8 and 9), and the small slope approximation, (10 and 11).

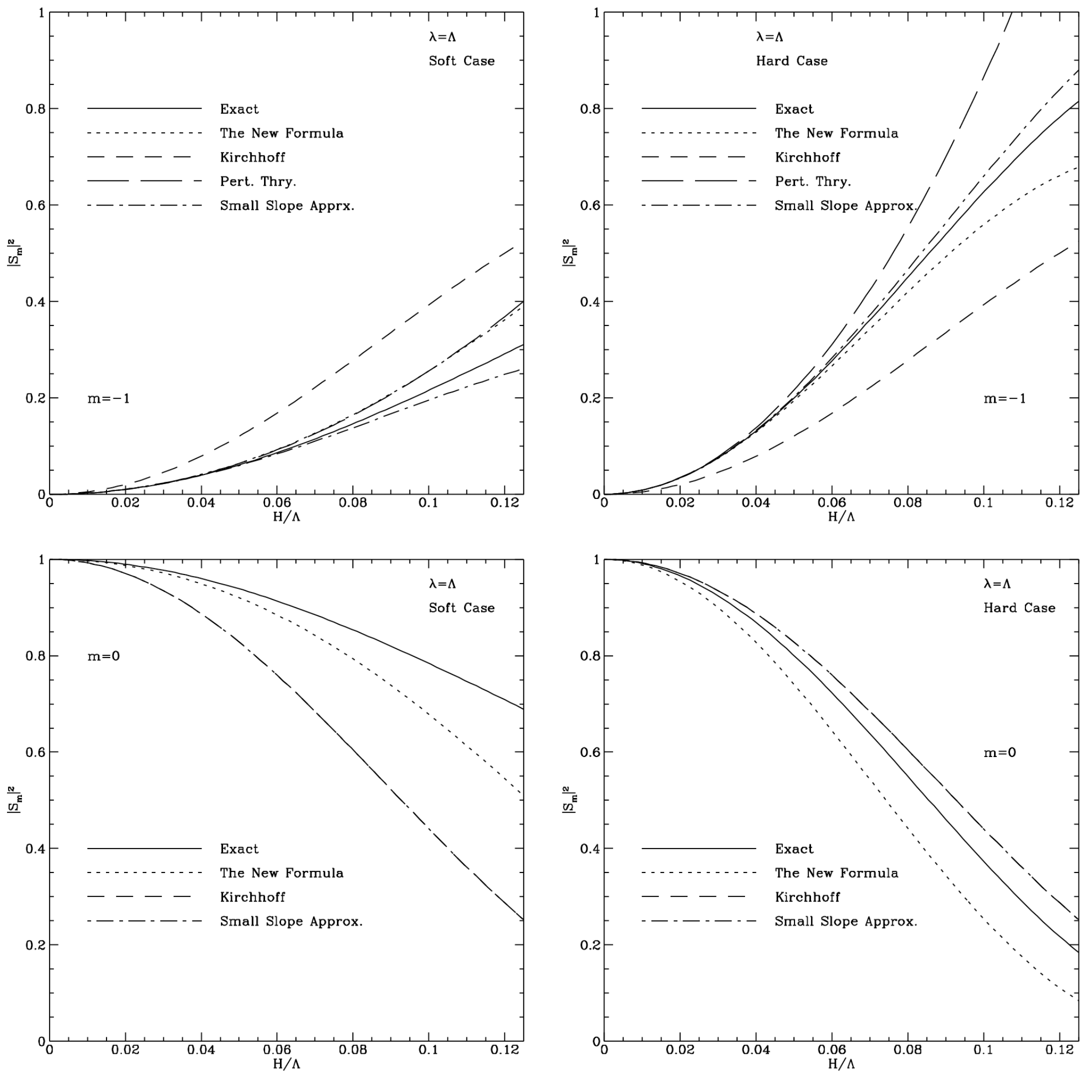

Figure 3: *Comparison of the scattering amplitude obtained from the new scattering amplitude formula and other standard formulas with the exact results for* $\lambda = \Lambda$ *and incident angle,* $\theta = 75$ *degrees. The direction of reflection is related to m according to Equation (5).*

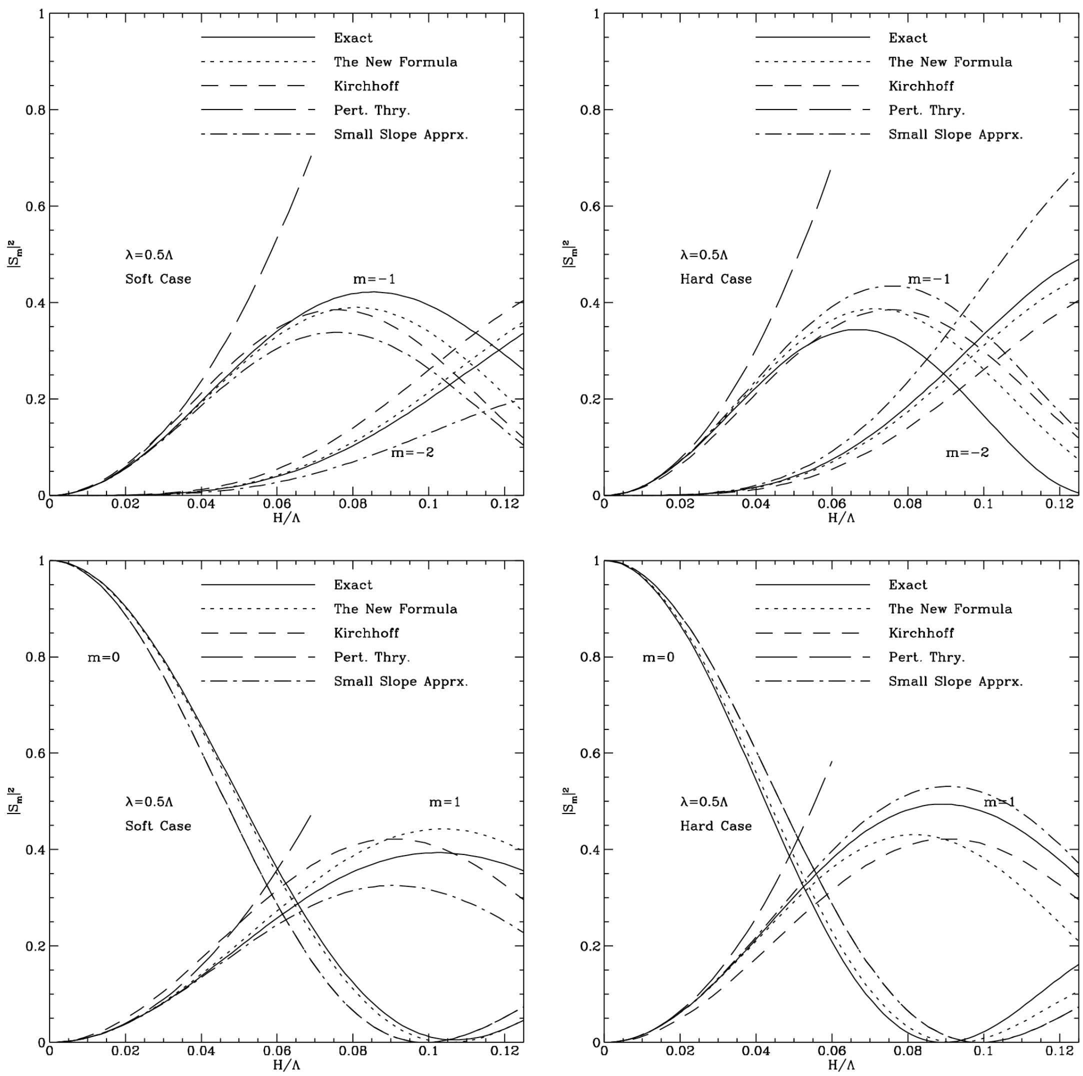

Figure 4: *Comparison of the scattering amplitude obtained from the new scattering amplitude formula and other standard formulas with the exact results for $\lambda = 0.5\Lambda$ and incident angle, $\theta = 75$ degrees. The direction of reflection is related to m according to Equation (5).*

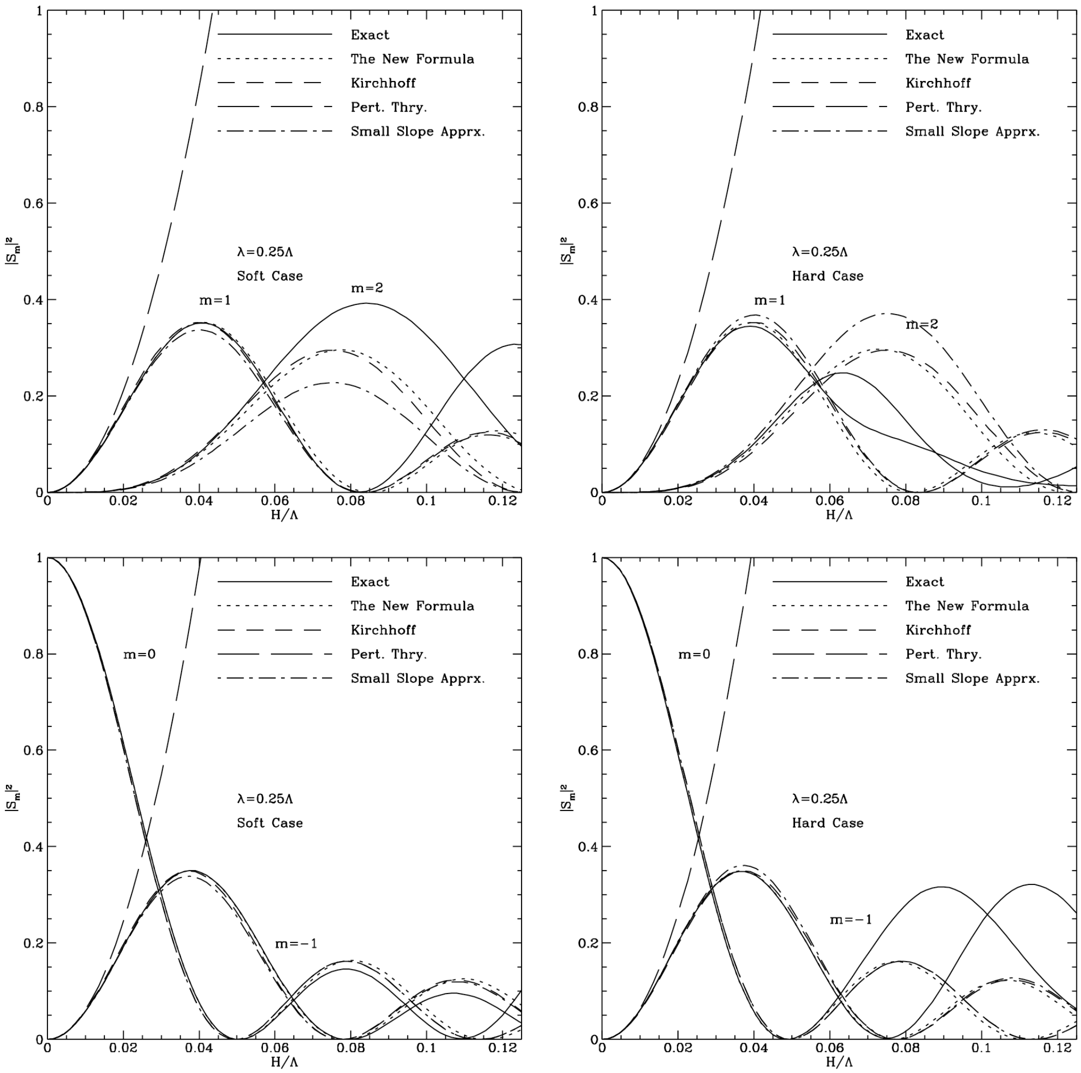

Figure 5: *Comparison of the scattering amplitude obtained from the new scattering amplitude formula and other standard formulas with the exact results for $\lambda = 0.25\Lambda$ and incident angle, $\theta = 75$ degrees. The direction of reflection is related to m according to Equation (5).*

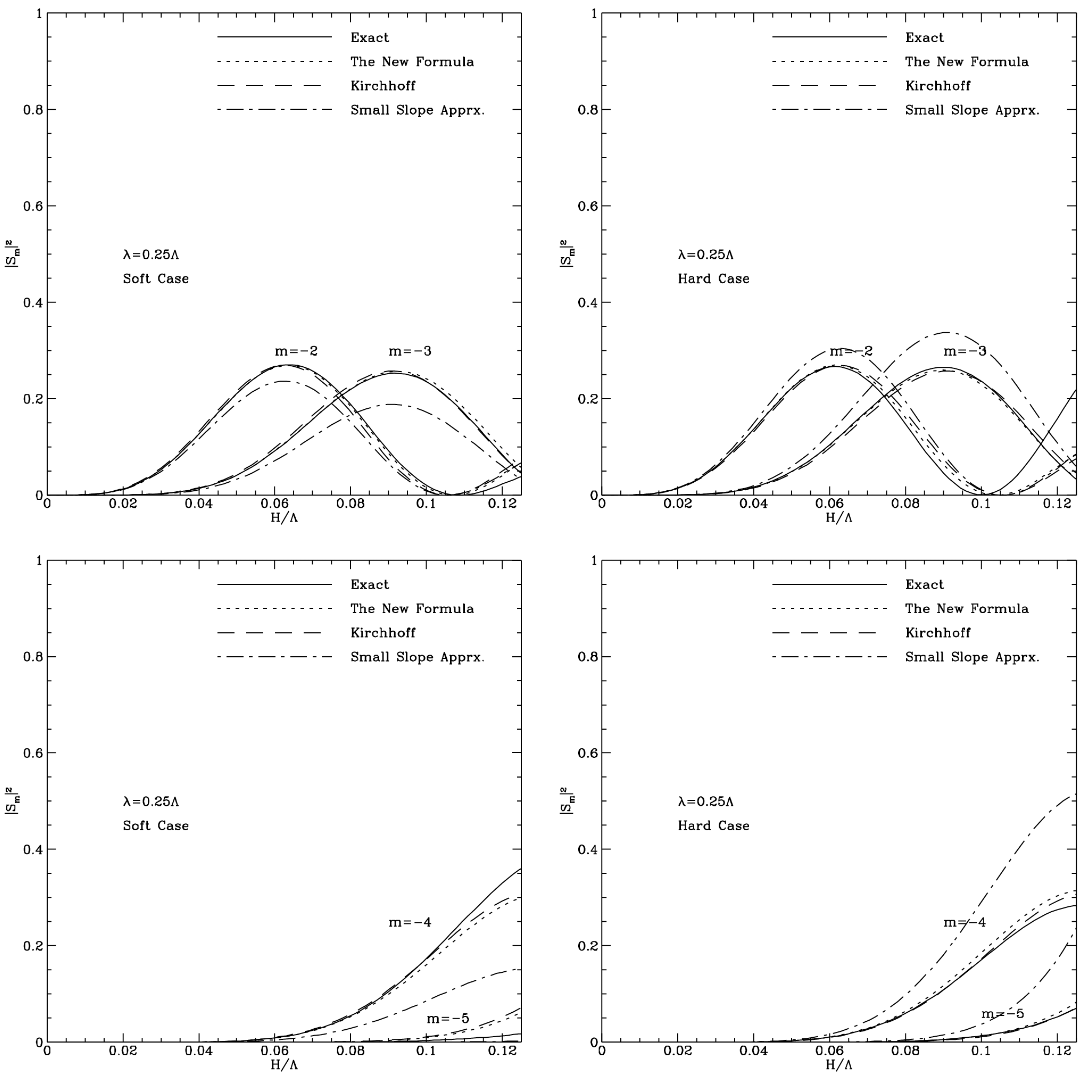

Figure 6: *Comparison of the scattering amplitude obtained from the new scattering amplitude formula and other standard formulas with the exact results for $\lambda = 0.25\Lambda$ and incident angle, $\theta = 75$ degrees. The direction of reflection is related to m according to Equation (5).*

Conclusions

The proceeding plots demonstrate the close agreement between our results, (6), and the exact results. It can be seen that these results are more accurate than those of Kirchhoff's (physical optics) and the small slope approximation in all regimes. Furthermore, it is observed that unlike Kirchhoff's solution, our solution approaches that of small perturbation solution, (8,9), for small surface heights. To save space, plots for angles of incidence other than 75 degrees are not presented here. However, we found that even though the agreement between our results and the exact results deteriorates as the angle of incidence becomes smaller, this agreement is better than those obtained by the other three methods considered here. The point that needs to be emphasized is that: as it is shown by the above plots, for small surface heights, when perturbation theory is valid, all the above results except for Kirchhoff's agree well with the exact results . On the other hand at high frequencies and larger surface heights only our results and Kirchhoff's have close agreement with the exact results as long as there is no shadowing and/or multiple scattering. Our formula thus forms a bridge between perturbation theory and small slope approximation in one limit and Kirchhoff's approximation in the other limit; and remains valid throughout the whole range and beyond where they are valid. For large ratios of surface amplitude to surface wavelength all of the above results are seen to deviate from exact results. Our formulation like Kirchhoff's, small slope approximation and perturbation theory does not account for non-local effects like shadowing and multiple scattering and is not expected to agree with the exact results when these effects are present. However, these effects do not begin to occur until the ratio of the surface amplitude to the surface wavelength is about 0.2. The question why all the approximate results start to deviate from the exact results even before shadowing and multiple scattering are present is an issue that needs to be investigated further.

References

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