The use of the virtual source technique in computing scattering from periodic ocean surfaces

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In this paper the virtual source technique is used to compute scattering of a plane wave from a periodic ocean surface. The virtual source technique is a method of imposing boundary conditions using virtual sources, with initially unknown complex amplitudes. These amplitudes are then determined by applying the boundary conditions. The fields due to these virtual sources are given by the environment Green's function. In principle, satisfying boundary conditions on an infinite surface requires an infinite number of sources. In this paper, the periodic nature of the surface is employed to populate a single period of the surface with virtual sources and *m* surface periods are added to obtain scattering from the entire surface. The use of an accelerated sum formula makes it possible to obtain a convergent sum with relatively small number of terms (∼40). The accuracy of the technique is verified by comparing its results with those obtained using the integral equation technique.
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I. INTRODUCTION

The problem of scattering of waves from periodic surfaces, especially sinusoidal surfaces has attracted a fair amount of interest. One of the reasons for this is that the simple nature of the boundary surface offers rather simple mathematical treatment; the other reason is that scattering from a periodic surface results in strong scattering in directions other than specular. Such non-specular returns are important in studying scattering from traveling ocean waves in underwater acoustics and from diffraction gratings in optics. This problem was first treated by Lord Rayleigh[1] who used the periodic nature of the surface and the boundary conditions on it to reduce the problem to a set of algebraic equations for the so-called reflection coefficients. He then proceeded to find these coefficients by an approximate method. Throughout the years many authors have treated this problem by using approximate methods like perturbation theory, when the surface height is small, and physical optics (Kirchhoff approximation), when the wavenumber is large compared with the curvature of the surface.

Uretsky[2] was the first to provide a solution based on the Helmholtz-Kirchhoff integral, but like Rayleigh's solution his solution fails as a result of some inadequacies, which do not seem to be understood or appreciated. According to Holford,[3] the problem centers on the inversion of infinite systems of algebraic equations. The first rigorous solution of a related problem of scattering from a periodic surface with impedance boundary conditions was provided by Urusovskii.[4–6] However, the solution for a pressure-release boundary condition can only be obtained from Urusovskii's solution by a limiting procedure, which reduces to Uretsky's solution and has similar problems. The first rigorous solution of scattering of a plane wave from a pressure-release boundary condition (Dirichlet boundary condition) was provided by Holford,[3] who solved the Helmholtz-Kirchhoff integral equation for the unknown pressure and its normal derivative on the boundary surface. The application of the integral equation technique to an infinite boundary results in an infinite set of algebraic equations. Uretsky solved these equations by the method of reduction, which is essentially a process of truncation and matrix inversion. However, according to Holford this method is only valid for systems of algebraic equations resulting from integral equations of the second kind. For systems of algebraic equations resulting from integral equations of the first kind, which Uretsky applied this technique to, no theoretical justification exists. That is, there is no guarantee that the solution obtained this way converges as the number of equations in the system is increased nor is there any guarantee that the solution converges to the correct solution. Holford solved these equations by the same method, but in doing so he had to formulate the problem in terms of an integral equation of the second kind.

In this paper, we use the virtual source technique to provide an exact solution for scattering of a plane wave from a periodic ocean surface. The virtual source technique, also known as the method of superposition, is a method of imposing boundary conditions on a surface (or interface) by using virtual sources of unknown complex amplitudes, which are determined by applying the boundary conditions. The functions representing the fields due to the virtual sources must satisfy the wave equation and the radiation condition. For this purpose, it is convenient to mathematically represent them by the environment Green's functions.

The virtual source method has widely been used in target scattering computations, particularly when the target is located in a waveguide.[7–10] And more recently, it has been used to model just the propagation in complicated

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waveguides[11,12] without a target. In this paper we apply the virtual source technique to an infinite periodic surface satisfying the pressure-release or Dirichlet boundary condition. Because of the infinite extent of the boundary surface, in principle an infinite number of sources are required to satisfy the boundary condition. However, it turns out that the very periodic nature of the boundary enables the solution to be constructed over one period. But to obtain the solution for the entire boundary, an infinite number of such solutions must be added. This construction keeps the number of unknowns, and consequently the number of equations the same as the number of virtual sources, n, in a single period.

We seek convergence by determining how many surface periods should be summed. The quantity inside the sum is a $n \times n$ matrix containing free space Green's functions. We use an accelerated sum formula, which converges to the correct solution for relatively small ($\sim$40) number of terms. The convergence properties of this formula are established in Ref. 13. In this regard our approach is very different from those of Uretsky and Holford, who seek convergence by determining how many algebraic equations should be used. The virtual source technique provides a rather simple solution to the problem of scattering from a periodic surface, as it only uses the free space Green's functions to construct the solution.

The outline of this paper is as follows. In Sec. 2, the virtual source solution is derived. In Sec. 3, the nature of the solution of the wave equation for a periodic surface is discussed and the method of obtaining the solution using the virtual source technique is described. In Sec. 4, the method is applied to a sinusoidal surface and its validity is demonstrated by comparing its results with those of Holford's integral equation solution. The paper is concluded in Sec. 5.

II. THE VIRTUAL SOURCE SOLUTION FOR A PERIODIC SURFACE

The geometry of the problem is shown in Fig. 1, where a plane wave is incident on a pressure-release periodic surface. The surface, which is given by $z = \zeta(x)$, is uniform in the y direction. This problem is solved in two-dimensions and the solution is denoted by $p(x, z; k)$, where k is the free-space wavenumber. Because the boundary is independent of the y coordinate, it follows immediately that for a non-penetrable surface the solution in three-dimensions can be obtained from the 2-D solution by

$$p(x, y, z; k) = e^{iky\sin\beta} p(x, z; k\cos\beta),$$

where β is the angle that the incident wave makes with the xz plane.

Due to the periodic nature of the surface, the incident field is scattered exactly the same way from two points separated by a surface wavelength, Λ. If the path length difference between rays scattered from these two points is a multiple of the acoustic wavelength, λ, rays interfere constructively. This can be expressed by the well-known grating

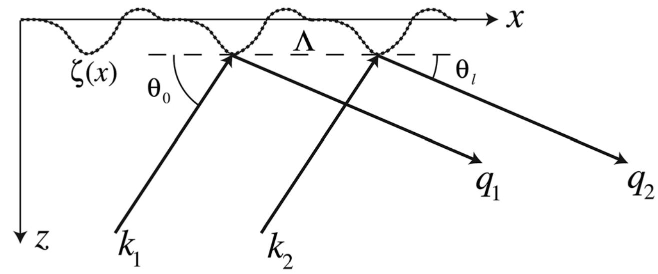

FIG. 1. The geometry for the scattering problem.

equation, which can be obtained by taking the path length difference between rays $k_1 q_1$ and $k_2 q_2$

$$\cos\theta_l = \cos\theta_0 + \frac{\lambda}{\Lambda} l. \quad (1)$$

In the above equation, θ_0 represents the direction of the incident wave and θ_l represents the directions for which the wave scattered from the individual periods of the surface will be in phase and will reinforce each other, giving rise to the modes of the scattering problem.

We will use the virtual source technique to impose the pressure-release boundary condition on the periodic surface S, represented by $\zeta(x)$. For this purpose we place virtual sources on a surface S', which is exactly like S, but translated slightly below it. The field due to each virtual source is represented by the environment Green's function with an unknown complex coefficient, which will be determined by satisfying the boundary condition at points on S, referred to as nodes and shown as dots on the boundary surface in Fig. (1). The Green's function for the environment is given by

$$G(\mathbf{r}; \mathbf{r}') = \frac{i}{4} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|), \quad (2)$$

where $\mathbf{r} = (x, z)$ is a field point, $\mathbf{r}' = (x', z')$ a source point, and $H_0^{(1)}$ is the Hankel function. The total field due to the incident field and the field produced by the virtual sources at point $\mathbf{r}$ is given by

$$\psi(\mathbf{r}) = \psi_{\text{inc}}(\mathbf{r}) + \sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} G\left(\mathbf{r}; \mathbf{r}'_{n+mN}\right) Q_{n+mN}. \quad (3)$$

Here N represents the number of virtual sources in one period of the surface, m represents the number of periods and Q_i represents the unknown complex source amplitude located at $\mathbf{r}'_i$. ψ_{inc} is the incident plane wave given by

$$\psi_{\text{inc}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ik(x\cos\theta_0 - z\sin\theta_0)}. \quad (4)$$

To determine the coefficients Q, we apply the pressure-release boundary condition $\psi\left(\mathbf{r}_p \in S\right) = 0$. Then Eq. (3) becomes,

$$\psi_{\text{inc}}[x_p, \zeta(x_p)] + \sum_{n=1}^{N} \sum_{m=-\infty}^{\infty} G[x_p, \zeta(x_p); x'_{n+mN}, \zeta'(x'_{n+mN})] \times Q_{n+mN} = 0, \quad (5)$$

where $\mathbf{r}'_n \in S'$. Since the surface S is periodic with period Λ, $\zeta(x+\Lambda)=\zeta(x)$, the incident field has the property

$$\psi_{\rm inc}[x+\Lambda, \zeta(x+\Lambda)] = e^{ik\Lambda\cos\theta_0}\psi_{\rm inc}[x,\zeta(x)], \quad (6)$$

which shows that it must be multiplied by $e^{ik\Lambda\ \cos\theta_0}$ as one moves one surface wavelength to the right. Similarly, since S' is also periodic with period $\Lambda, x'_{n+mN} = x'_n + m\Lambda$ and $\zeta'(x'_{n+mN}) = \zeta'(x'_n + m\Lambda) = \zeta'(x'_n)$. Due to the infinite, periodic nature of the surface, the source amplitudes for two sources a distance Λ apart must also satisfy

$$Q[x'+\Lambda, \zeta'(x'+\Lambda)] = e^{ik\Lambda\cos\theta_0}Q[x', \zeta'(x')].$$

Using this property, we can write

$$Q_{n+mN} = e^{imk\Lambda\cos\theta_0}Q_n.$$

Using these arguments, Eq. (5) can be written as

$$\psi_{\rm inc}[x_p, \zeta(x_p)] + \sum_{n=1}^{N}\Gamma^m_{pn}Q_n = 0, \quad (7)$$

where

$$\Gamma^m_{pn} = \sum_{m=-\infty}^{\infty} G\left[x_p, \zeta(x_p); x'_n + m\Lambda, \zeta\left(x'_n\right)\right]e^{imk\Lambda\cos\theta_0}. \quad (8)$$

Let the position of all the nodes on the surface be denoted by a vector $\mathbf{r} = [\mathbf{x}, \zeta(\mathbf{x})]$ of length n, and the position of sources be denoted by a vector of the same length $\mathbf{r}' = [\mathbf{x}', \zeta(\mathbf{x}')]$. From Eq. (7) the source amplitudes can be obtained from the following equation written in vector-matrix notation

$$\mathbf{Q} = -[\Gamma^m]^{-1}\psi_{\rm inc}. \quad (9)$$

In the above equation $\psi_{\rm inc}$ is a column vector of length n containing the incident field at n nodal points, $\mathbf{Q}$ is a column vector of length n containing the complex source amplitudes and $\mathbf{\Gamma}^m$ is an $n \times n$ matrix. Substitution of $\mathbf{Q}$ in Eq. (3) gives the field everywhere in the medium. The Green's function in Eq. (8) is given by Eq. (2), which gives

$$\Gamma^m_{pn} = \frac{i}{4}\sum_{m=-\infty}^{\infty} H_0^{(1)}\left(kr^m_{pn}\right)e^{imk\Lambda\cos\theta_0}, \quad (10)$$

where

$$r^m_{pn} = \sqrt{\left[x_p - \left(x'_n + m\Lambda\right)\right]^2 + \left(\zeta_p - \zeta'_n\right)^2}.$$

It is not practical to compute Γ using Eq. (10) since the sum over the Hankel function converges very slowly and requires m to be very large. To deal with this, we derive an accelerated sum formula below.

A. An accelerated sum formula for Eq. (10)

The formula that we are about to derive is due to Linton,[13] which is included here for completeness. Consider the sum

$$G(X,Z) = \frac{i}{4}\sum_{m=-\infty}^{\infty} H_0^{(1)}(kr_m)e^{im\beta d}, \quad (11)$$

where

$$X = x_p - x'_n, \quad Z = \zeta_p - \zeta'_n, \quad \beta = k\cos\theta_0$$

$$r_m = \sqrt{(X-md)^2 + Z^2}.$$

Equation (11) can be expressed as an accelerated sum by using the Poisson sum formula,

$$\sum_{m=-\infty}^{\infty} e^{imu} = 2\pi\sum_{m=-\infty}^{\infty}\delta(u + 2m\pi), \quad (12)$$

and an integral representation of the Hankel function (Ref. 13):

$$H_0^{(1)}\left(k\sqrt{a^2+b^2}\right) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\gamma^{-1}e^{-k\gamma|b|}e^{-ikat}dt, \quad (13)$$

where $\gamma = \sqrt{t^2-1} = i\sqrt{1-t^2}$. Using the above representation in Eq. (11) with $a = X - md$ and $b = Z$ gives

$$G(X,Z) = \frac{1}{2}\int_{-\infty}^{\infty}\gamma^{-1}e^{-k\gamma|Z|}e^{-iktX}\sum_{m=-\infty}^{\infty}e^{imd(kt+\beta)}dt, \quad (14)$$

where the sum and the integral signs have been reversed. According to the Poisson sum formula

$$\sum_{m=-\infty}^{\infty}e^{imd(kt+\beta)} = \sum_{m=-\infty}^{\infty}\delta[d(kt+\beta)+2m\pi].$$

Substituting this in Eq. (14), and again reversing the sum and the integral we get

$$G(X,Z) = \frac{1}{2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma^{-1}e^{-k\gamma|Z|}e^{-iktX}\delta[d(kt+\beta)+2m\pi]dt.$$

The sifting property of the δ function reduces the above equation to

$$G(X,Z) = \frac{1}{2d}\sum_{m=-\infty}^{\infty}\frac{e^{-\sqrt{(mp+\beta)^2-k^2}|Z|}}{\sqrt{(mp+\beta)^2-k^2}}e^{ikX(mp+\beta)},$$

where $p = 2\pi/d$. Letting $\beta_m = mp + \beta$ and $\gamma_m \equiv \sqrt{\beta_m^2 - k^2}$, we finally get the accelerated form of Eq. (10)

$$G(X,Z) = \frac{1}{2d}\sum_{m=-\infty}^{\infty}\frac{e^{-\gamma_m|Z|}e^{i\beta_m X}}{\gamma_m}. \quad (15)$$

The exponential in the above sum ensures its rapid convergence. Other properties of the above equation are discussed in Ref. 13.

III. THE NATURE OF THE SOLUTION OF THE WAVE EQUATION FOR A PERIODIC SURFACE

Scattering of a plane wave from a periodic surface can be described by the following expression due to Rayleigh:

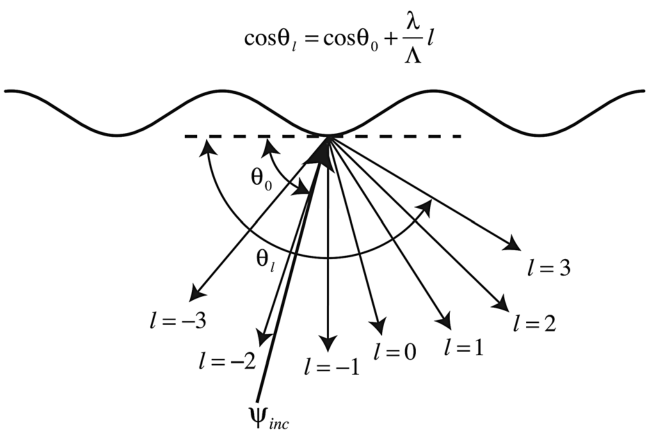

FIG. 2. Scattering of a plane wave from a periodic surface.

$$p(x,z) = e^{ik(\alpha_0 x - \gamma_0 z)} + \sum_{n=-\infty}^{\infty} R_n e^{ik(\alpha_n x + \gamma_n z)}, \tag{16}$$

where $\alpha_n = \cos\,\theta_n$ and $\gamma_n = \sin\,\theta_n$. According to the above equation (see Fig. 2), below the hollows of the boundary, the scattered field consists of discrete plane waves of amplitude R_n, a finite number of which (γ_n real) propagate from the boundary, while the remainder (γ_n purely imaginary) correspond to surface waves since their amplitudes decay exponentially in the z direction. According to Eq. (1) propagating solutions are obtained only when $\lambda/\Lambda \le 1$ and $l = 0$. This means that there is always a propagating wave in the specular direction ($l = 0$) regardless of the ratio of the radiation wavelength to the surface wavelength, λ/Λ. However, propagating waves in directions other than specular can only occur when $\lambda/\Lambda \le 1$. The coefficients R_n in Eq. (16) may be thought of as a set of transformation coefficients that carry the energy of the incident wave in the direction θ_0 into a set of outgoing waves in the directions θ_n. This concept can be used to define a scalar product and derive an expression for the conservation of energy for a surface with no attenuation (Ref. 2)

$$\sum_{\gamma_n\ real} \frac{\gamma_n}{\gamma_0} |R_n|^2 = 1, \tag{17}$$

where it is assumed that the amplitude of the incident wave, $R_0 = 1$. Rayleigh used Eq. (16) to solve the problem of scattering of a plane wave from a sinusoidal surface. He did this by setting the left hand side of the above equation to 0 on the boundary and employing the periodicity of the surface to obtain a set of algebraic equations from which he determined the first few coefficients R_n by successive substitutions. Rayleigh's solution, however, is only an approximation, as Eq. (16) is not valid within the hallows of the boundary, $\zeta(x) \le z < \zeta_{\rm maz}$, where one would expect to find both up- and down-going waves, rather than a single set of down-going waves as in Eq. (16). An interesting discussion on the validity of the Rayleigh's solution is given in Ref. 3.

The integral equation technique in Ref. 3 and the virtual source solution described in this paper, of course, solve the problem exactly and the solution is valid everywhere in the computational domain. These solutions also satisfy conservation of energy described by Eq. (17). The virtual source

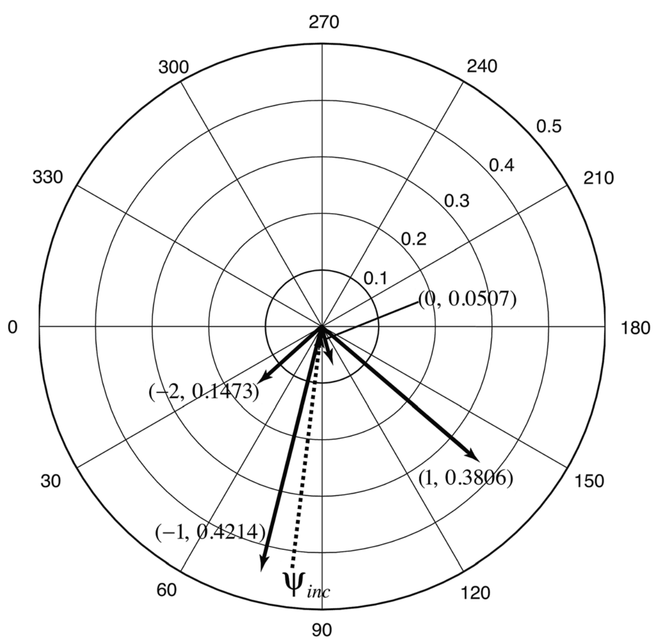

FIG. 3. This figure shows how energy is distributed among various propagating modes for a plane wave scattered from a sinusoidal surface. The numbers inside the brackets represent the order of the scattered mode followed by its relative amplitude, $\gamma_n/\gamma_0|R_n|^2$.

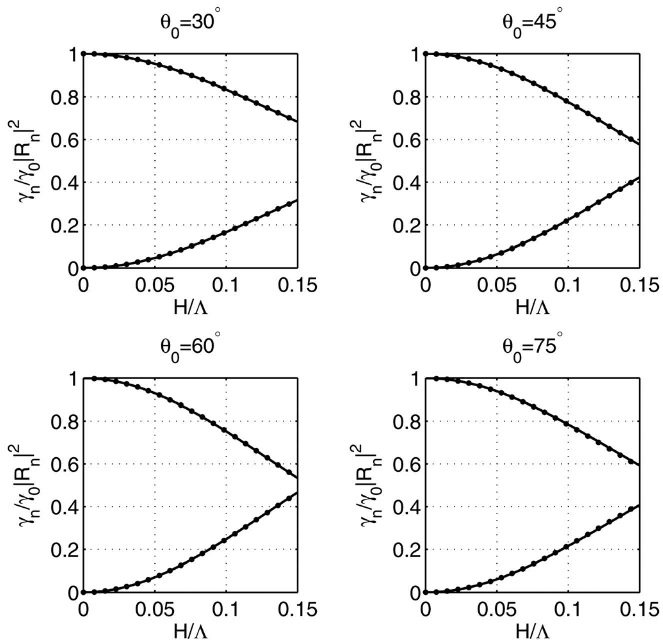

FIG. 4. Comparison between the integral equation solution (solid lines) and the virtual source solution (small circles) of scattering of a plane wave from a sinusoidal surface as function of the ratio of surface height to surface wavelength. In the above figure, the wavelength of the incident plane wave is equal to the surface wavelength, resulting in two propagating modes, $l = -1$ and $l = 0$. Each panel is for a different angle of incidence. The top curves represent the normalized intensity for mode $l = 0$, which corresponds to the specular reflection. Note that when the surface is flat ($H/\Lambda = 0$), the reflected energy is entirely in the specular direction. As the surface roughness increases, the off-specular mode, $l = -1$, begins to contribute to the total scattered energy. But at any surface roughness, the sum of the energy from these two modes is one, according to Eq. (17).

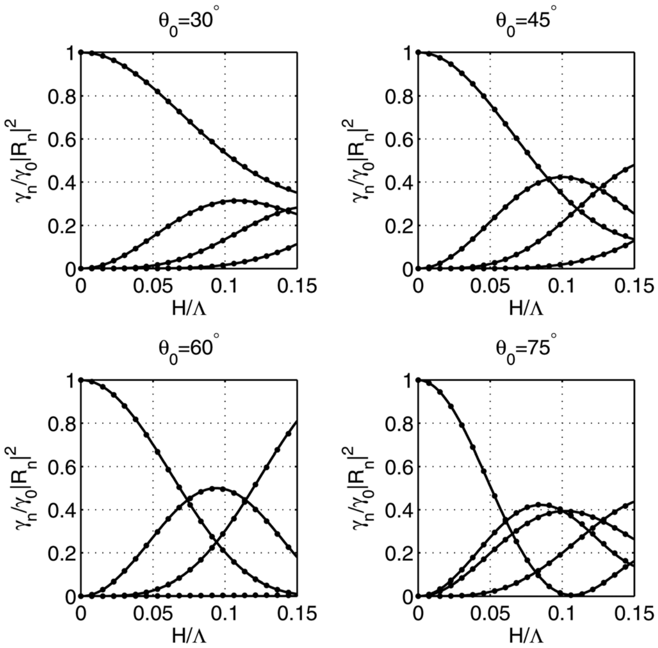

FIG. 5. Comparison between the integral equation solution (solid lines) and the virtual source solution (small circles) of scattering of a plane wave from a sinusoidal surface as function of the ratio of surface height to surface wavelength. In the above figure, the wavelength of the incident plane wave is half the surface wavelength, resulting in four propagating modes: $l=-3$, -2, -1,0 for incident angles 30, 45 and 60° and $l=-2$, -1, 0, 1 for incident angle 75°. Each panel is for a different angle of incidence. The curves that have values of one at $H/\Lambda=0$ correspond to the mode $l=0$. The other modes have not been labeled. Note that the off-specular modes begin to contribute to the total scattered energy as the surface roughness increases; and at any surface roughness the sum of the energies of all the propagating modes is one, according to Eq. (17).

method computes the total field, which is composed of all propagating modes. Each component of the propagating modes, R_n is computed by projecting the total field onto the propagating angles determined by Eq. (1). In Fig. (3) we computed $\gamma_n/\gamma|R_n|^2$ for the four propagating modes that result in scattering of a plane wave from a sinusoidal surface. In this case, the incident field is at 75° and the surface is represented by

$$\zeta(x) = 0.09\cos\left(\frac{2\pi x}{\Lambda}\right),$$

where in the above Λ, the surface wavelength is 2π and the wavelength of the incident field is 0.5Λ. The numbers inside the brackets in Fig. (3) represent the order of the scattered mode followed by its relative amplitude, $\gamma_n/\gamma_0|R_n|^2$. The sum of the relative amplitudes for all the modes equals 1 in accordance with Eq. (17).

IV. COMPARISON WITH THE INTEGRAL EQUATION SOLUTION

In this section we benchmark the solution of scattering of a plane wave from a sinusoidal surface against the integral equation solution obtained by R. Holford (private communication). The surface in these benchmark solution is given by

$$\zeta(x) = H\cos\left(\frac{2\pi x}{\Lambda}\right),$$

where H is the surface amplitude and Λ is its wavelength. In all of the computations reported here we use 60 sources per acoustic wavelength and use the projection method described in the previous section to compute the components of the propagating modes as a function of H/Λ. The latter is a measure of surface roughness and is used as an independent variable in these computations.

In Fig. (4) we show comparisons between the integral equation method and the virtual source method for each normalized propagating mode intensity. In this case $\lambda=\Lambda$ and the computations are carried out for four incident angles. For this ratio of the acoustic wavelength to surface wavelength, Eq. (1) allows two propagating modes: $l=-1$ and $l=0$. The top curves represent the normalized intensity for mode $l=0$, which corresponds to the specular reflection. Note that when the surface is flat ($H/\Lambda=0$), the reflected energy is entirely in the specular direction. As the surface roughness increases, the off-specular mode, $l=-1$, begins to contribute to the total scattered energy. But at any surface roughness, the sum of the energy from these two modes is one, according to Eq. (17).

In Fig. 5 we repeat the same computations as in Fig. 4, but for higher frequency. In the this case the wavelength of the incident plane wave is half of the surface wavelength, resulting in four propagating modes, $l=-3,\ -2,\ -1,0$ for incident angles 30°, 45°, and 60° and $l=-2,\ -1,\ 0,1$ for

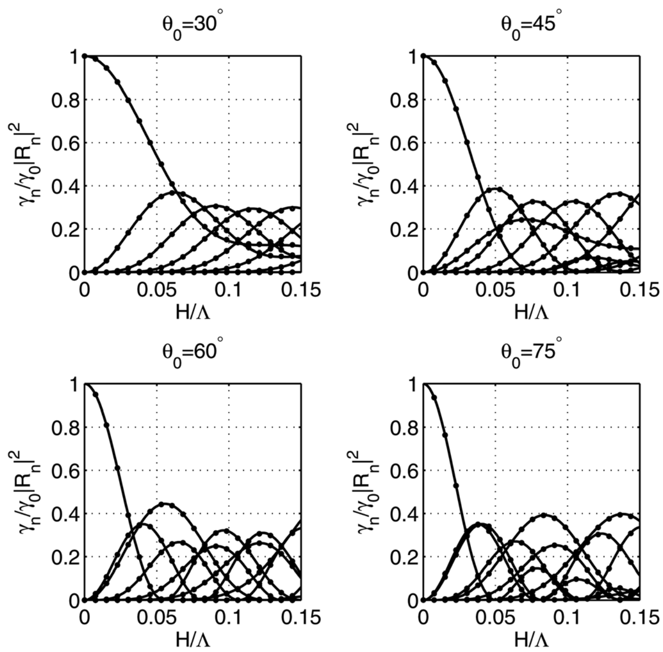

FIG. 6. Comparison between the integral equation solution (solid lines) and the virtual source solution (small circles) of scattering of a plane wave from a sinusoidal surface as function of the ratio of surface height to surface wavelength. In the above figure, the wavelength of the incident plane wave is a quarter of the surface wavelength, resulting in eight propagating modes: $l=-7,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1$, 0 for an incident angle of 30°, $l=-6,\ -5,\ -4,\ -3,\ -2,\ -1$, 0, 1 for incident angles of 45° and 60° and $l=-5,\ -4,\ -3,\ -2,\ -1$, 0, 1, 2 for an incident angle of 75°. The curves that have values of one at $H/\Lambda=0$ correspond to the mode $l=0$. The other modes have not been labeled. Note that the off-specular modes begin to contribute to the total scattered energy as the surface roughness increases; and at any surface roughness the sum of the energies of all the propagating modes is one, according to Eq. (17).

incident angle 75°. Note that the off-specular modes begin to contribute to the total scattered energy as the surface roughness increases; and at any surface roughness the sum of the energies of all the propagating modes is one, according to Eq. (17).

In Fig. (6), we repeat the computations for the case when $\lambda = 0.25\Lambda$. in this case, Eq. (1) allows eight propagating modes: $l = -7, -6, -5, -4, -3, -2, -1, 0$ for an incident angle of 30°, $l = -6, -5, -4, -3, -2, -1, 0, 1$ for incident angles of 45° and 60° and $l = -5, -4, -3, -2, -1, 0, 1, 2$ for an incident angle of 75°. Again, the contribution from off-specular angles start when H/Λ is non-zero. However, conservation of energy is satisfied for all values of H/Λ. In the above figures, individual modes have not been labeled to save space. However, our goal in presenting these results is to show that the virtual source solution produces identical results to those obtained from the integral equation solution for all propagating modes and a wide range of incident angles.

V. CONCLUSIONS

In this paper we solved the problem of scattering of a plane wave from a periodic surface satisfying the pressure-release boundary condition using the virtual source technique. The invariance of the surface in the y coordinate allowed us to solve a three-dimensional scattering problem in two-dimensions and thus mathematically represented the virtual sources by the two-dimensional, free space Green's function with unknown complex amplitudes. The amplitudes of the virtual sources were determined from the boundary conditions.

In principle, satisfying boundary conditions on an infinite surface requires an infinite number of sources. In this paper, we employed the periodic nature of the surface to populate a single period of the surface with virtual sources and added m surface periods to obtain scattering from the entire surface. The use of an accelerated sum formula enabled us to obtain a convergent sum with relatively small number of terms ($\sim$40).

We applied this technique to compute scattering from a sinusoidal surface as a function of surface amplitude to surface wavelength for a variety of incident angles and frequencies, λ/Λ. To demonstrate the accuracy of our solution, we compared our solutions with those obtained by R. Holford (private communication) using the integral equation technique and found out that the two solutions are essentially identical for all cases considered.

The technique described in this paper provides a simple way to obtain an independent benchmark solution for computing scattering from a periodic pressure-release surface. Extension of the virtual source technique to compute scattering from a rigid periodic surface is a topic to be explored in the future.

ACKNOWLEDGMENTS

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[1]L. Rayleigh, *The Theory of Sound*, Vol. 2 (Dover Publications, New York, 1945), pp. 89–96.
[2]J. L. Uretsky, "The scattering of plane waves from periodic surface," Ann. Phys. Leipzig **33**, 400–427 (1965).
[3]R. Holford, "Scattering of waves at a periodic pressure release surface: An exact solution," J. Acoust. Soc. Am. **70**(4), 1116–1128 (1981).
[4]I. A. Urusovoskii, "Sound scattering by a sinusoidally uneven surface characterized by normal acoustic conductivity," Sov. Phys. Acoust. **5**, 362–369 (1959).
[5]I. A. Urusovoskii, "Diffraction by a periodic surface," Sov. Phys. Acosut. **10**, 287–293 (1965).
[6]I. A. Urusovoskii, "Diffraction of sound on a periodically uneven and inhomogeneous surface," Sov. Phys. Dokl. **5**, 345–348 (1960).
[7]P. R. Stepanishen, "A generalized internal source density method for the forward and backward projection of harmonic pressure fields from complex bodies," J. Acoust. Soc. Am. **101**(6), 3270–3277 (1997).
[8]G. H. Koopman, L. Song, and J. B. Fahnline, "A method for computing acoustic fields based on the principle of wave superposition," J. Acoust. Soc. Am. **86**(6), 2433–2438 (1989).
[9]J. B. Fahnline and G. H. Koopman, "A numerical solution for the general radiation problem based on the combined methods of superposition and singular-value decomposition," J. Acoust. Soc. Am. **90**(5), 2808–2819 (1991).
[10]M. Ochmann, "The source simulation technique for acoustic radiation problems," Acustica **81**, 512–527 (1995).
[11]J. A. Fawcett, "A scattering-chamber approach for solving finite rough surface scattering problems," J. Acoust. Soc. Am. **118**(3), 1348–1357 (2005).
[12]A. T. Abawi and M. B. Porter, "Propagation in an elastic wedge using the virtual source technique," J. Acoust. Soc. Am. **121**(3), 1374–1382 (2007).
[13]C. M. Linton, "The green's function for the two-dimensional helmholtz equation in periodic domains," J. Eng. Mech. **33**, 377–402 (1998).