Determination of the Resonant Frequencies of a Cone-Sphere by the Method of Moments

Ahmad T. Abawi
Hughes Research Laboratories, 3011 Malibu Canyon Road, Malibu, CA 90265

October 23, 2004

A cone-sphere is the object shown below. Its resonant frequencies or eigenfrequencies are the eigenvalues of the wave equation satisfying a given boundary conditions on the surface. Here, we are interested in solving the wave equation for Dirichlet boundary conditions; in other words we want the solution to vanish on the surface. A complicated geometry such as a cone-sphere makes it impossible to separate the wave equation and solve it by the method of separation of variables. We thus have to turn to numerical techniques. One numerical method which is relevant to this problem is the Galerkin method which is described at the end of this report. Since the cone sphere has cylindrical symmetry, let us write the wave equation in cylindrical coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + \kappa^2 \psi = 0 \tag{1}$$

The cone-sphere can be described by

$$q(z) = \begin{cases} \sqrt{2az - z^2}, & \text{for } z \leq D; \\ (H - z)\Omega, & \text{for } z \geq D. \end{cases}$$

where

$$\begin{aligned} \Omega &= \tan\alpha, \\ D &= a(1 + \sin\alpha), \\ H &= a(1 + \csc\alpha). \end{aligned}$$

Let us define a basis function which vanishes on the surface.

$$\psi_{j,k,m} = \sum_{j,k,m} A_{j,k,m} r^j (q(z)^2 - r^2) z^k e^{im\phi} \tag{2}$$

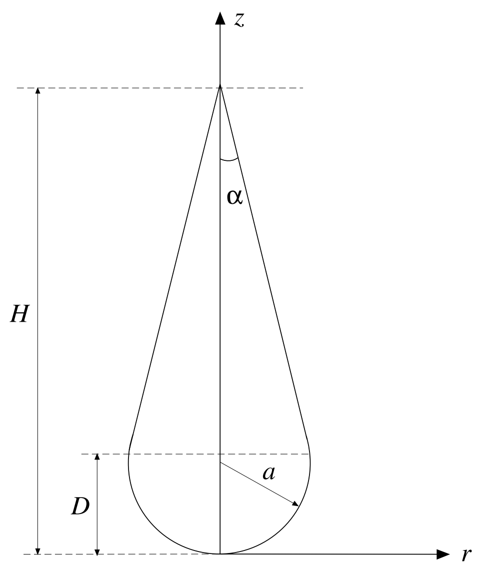

Figure 1: A Cone-Sphere

for

$$\begin{aligned} m &= 0, 1, .., \\ k &= 0, 1, .., \\ j &= m, m+1, ... \end{aligned}$$

where the coefficients $A_{j,k,m}$ are to be determined. According to the Galerkin method, we have

$$< \psi_{j',k',m'} | \bigtriangledown^2 | \psi_{j,k,m} > + < \psi_{j',k',m'} | \kappa^2 | \psi_{j,k,m} >= 0 \tag{3}$$

where

$$\psi_{j',k',m'} = \sum_{j',k',m'} r^{j'} (q(z)^2 - r^2) z^{k'} e^{im'\phi}$$

or

$$\sum \int (\psi_{j',k',m'} \bigtriangledown^2 \psi_{j,k,m} + \psi_{j',k',m'} \kappa^2 \psi_{j,k,m}) dv = 0, \tag{4}$$

where the above sum is over all indices. Since $\psi = 0$ on the surface, according to Green's theorem

$$\int \phi \bigtriangledown^2 \psi dv = - \int \bigtriangledown \phi \cdot \psi dv + \int_S \psi n \cdot \bigtriangledown \phi dS$$

Eq.(4) reduces to

$$\sum \int (\bigtriangledown \psi_{j',k',m'} \cdot \bigtriangledown \psi_{j,k,m}) dv = \sum \int (\psi_{j',k',m'} \kappa^2 \psi_{j,k,m}) dv = 0 \tag{5}$$

The integration over ϕ yields a delta function which requires that $m = m'$, then we are left with

$$\sum_{j,k} A_{j,k} \int (\bigtriangledown \psi_{j',k'} \cdot \bigtriangledown \psi_{j,k}) r dr dz = \sum_{j,k} \int \kappa^2 (\psi_{j',k'} \psi_{j,k}) r dr dz \tag{6}$$

for all j' and k'.
where

$$\psi_{j,k} = r^j (q(z)^2 - r^2) z^k$$

and

$$\psi_{j',k'} = r^{j'} (q(z)^2 - r^2) z^{k'}$$

letting

$$\begin{aligned} x &\equiv j + j', \\ y &\equiv k + k'. \end{aligned}$$

we have

$$I_{left} = \int_0^H \int_0^{q(z)} (\bigtriangledown \psi_{j',k'} \cdot \bigtriangledown \psi_{j,k}) r dr dz$$

and

$$I_{right} = \int (\psi_{j',k'} \psi_{j,k}) r dr dz$$

the r integration yields

$$I_{left} = \int_0^H \left\{ \frac{z^y q^{\frac{x}{2}} u_1}{x} + \frac{z^y q^{\frac{x+2}{2}} u_2}{x+2} + \frac{z^y q^{\frac{x+4}{2}} u_3}{x+4} + \frac{z^y q^{\frac{x+6}{2}} u_4}{x+6} \right\} dz$$

where

$$\begin{aligned} u_1 &= r^{-2} (q^2 (jj' + m^2)), \\ u_2 &= -2q(jj' + x + m^2) + kk' z^{-2} q^2 + y z^{-1} q \frac{dq}{dz} + (\frac{dq}{dz})^2, \\ u_3 &= (j+2)(j'+2) + m^2 - 2kk' q z^{-2} - y z^{-1} \frac{dq}{dz}, \\ u_4 &= kk' z^{-2} \end{aligned}$$

The integral I_{left} can be split into four integrals

$$I_{left} = I_1 + I_2 + I_3 + I_4$$

where,

$$
\begin{aligned}
I_1 &= \int_0^H z^y q^{\frac{x+4}{2}} \left\{ \frac{jj' + m^2}{x} - \frac{2(jj' + x + m^2)}{x+2} + \frac{(j+2)(j'+2) + m^2}{x+4} \right\} dz, \\
I_2 &= \int_0^H z^{y-2} y^{\frac{x+6}{2}} \left\{ \frac{kk'}{x+2} - \frac{2kk'}{x+4} + \frac{kk'}{x+6} \right\} dz, \\
I_3 &= \int_0^H z^{y-1} q^{\frac{x+4}{2}} \frac{dq}{dz} \left(\frac{y}{x+2} - \frac{y}{x+4} \right) dz, \\
I_4 &= \int_0^H z^y \frac{q^{\frac{x+2}{2}}}{x+2} \left(\frac{dq}{dz} \right)^2 dz.
\end{aligned}
$$

and we find

$$
I_{left} = \int_0^H \{ \frac{1}{j+j'+2} - \frac{2}{j+j'+4} + \frac{1}{j+j'+6} \} z^{k+k'} q^{\frac{j+j'+6}{2}} dz
$$

All of the above integrals turn out to be incomplete β functions. The incomplete β function is defined

$$
I_x(a,b) \equiv \frac{B_x(a,b)}{B(a,b)} \equiv \int_0^x t^{a-1}(1-t)^{b-1} dt
$$

a and b positive

$$
B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt
$$

is the complete β function. Defining

$$
\beta(a,b,x) \equiv B(a,b) I_x(a,b),
$$

we can write all of the above integrals in terms of $\beta(a,b,x)$.
First, let us define the following parameters

$$
\begin{aligned}
c_1 &= \frac{jj' + m^2}{x} - \frac{2(jj' + x + m^2)}{x+2} + \frac{(j+2)(j'+2) + m^2}{x+4}, \\
\beta_1 &= \frac{x+4}{2}, \\
\alpha_1 &= y, \\
c_2 &= \frac{kk'}{x+2} - \frac{2kk'}{x+4} + \frac{kk'}{x+6}, \\
\beta_2 &= \frac{x+6}{2}, \\
\alpha_2 &= y - 2, \\
c_3 &= y\left(\frac{1}{x+2} - \frac{1}{x+4} \right),
\end{aligned}
$$

$$
\begin{aligned}
\beta_3 &= \frac{x+4}{2}, \\
\alpha_3 &= y-1, \\
c_4 &= \frac{1}{x+4}, \\
\beta_4 &= \frac{x+2}{2}, \\
\alpha_4 &= y, \\
c_5 &= \frac{1}{j+j'+2} - \frac{2}{j+j'+4} + \frac{1}{j+j'+6}, \\
\beta_5 &= k+k', \\
\alpha_5 &= \frac{j+j'+6}{2}.
\end{aligned}
$$

Noting that integrals of the form

$$\int_0^H c_1 z^y q^{\frac{x+4}{2}} dz$$

can be written as

$$\int_0^H c_1 z^y q^{\frac{x+4}{2}} dz = \int_0^D c_1 z^y q_<^{\frac{x+4}{2}} dz + \int_D^H c_1 z^y q_>^{\frac{x+4}{2}} dz$$

where $q_<$ and $q_>$ denote the form the function $q(z)$ in the regions $z < D$ and $z > D$, respectively. The second integral in the above expression can be written

$$\int_D^H c_1 z^y q_>^{\frac{x+4}{2}} dz = \int_0^H c_1 z^y q_>^{\frac{x+4}{2}} dz - \int_0^D c_1 z^y q_>^{\frac{x+4}{2}} dz.$$

Skipping a fair amount of algebra, the above integrals are evaluated to be

$$
\begin{aligned}
I_1 &= c_1\{(2a)^{2\beta_1+\alpha_1+1}\gamma_1 + \Omega^{2\beta_1} H^{2\beta_1+\alpha_1+1}(\xi_1 - \zeta_1)\}, \\
I_2 &= c_2\{(2a)^{2\beta_2+\alpha_2+1}\gamma_2 + \Omega^{2\beta_2} H^{2\beta_2+\alpha_2+1}(\xi_2 - \zeta_2)\}, \\
I_3 &= c_3\{(2a)^{2\beta_3+\alpha_3+2}(\gamma_3 - 2\omega_3) + 2\Omega^{2\beta_3+2} H^{2\beta_3+\alpha_3+2}((\theta_3 - \nu_3) - (\xi_3 - \zeta_3))\}, \\
I_4 &= c_4\{(2a)^{2\beta_4+\alpha_4+3}(\gamma_4 - 4\omega_4 + 4\mu_4) \\
&+ 4\Omega^{2\beta_4+4} H^{2\beta_4+\alpha_4+3}((\xi_4 - \zeta_4) - 2(\theta_4 - \nu_4) + (\eta_4 - \sigma_4))\}, \\
I_{right} &= c_5(2a)^{2\beta_5+\alpha_5+1}\gamma_5 + \Omega^{2\beta_5} H^{2\beta_5+\alpha_5+1}(\xi_5 - \zeta_5)
\end{aligned}
$$

Where

$$
\begin{aligned}
\gamma_i &= \beta(\alpha_i + \beta_i + 1, \beta_i + 1, \frac{D}{2a}), \\
\xi_i &= \beta(\alpha_i + 1, 2\beta_i + 1, 1), \\
\zeta_i &= \beta(\alpha_i + 1, 2\beta_i + 1, \frac{D}{H}),
\end{aligned}
$$

$$
\begin{aligned}
\omega_i &= \beta(\alpha_i+\beta_i+2, \beta_i+1, \frac{D}{2a}), \\
\theta_i &= \beta(\alpha_i+2, 2\beta_i+1, 1), \\
\nu_i &= \beta(\alpha_i+2, 2\beta_i+1, \frac{D}{2a}), \\
\mu_i &= \beta(\alpha_i+\beta_i+3, \beta_i+1, \frac{D}{2a}), \\
\eta_i &= \beta(\alpha_i+3, 2\beta_i+1, 1), \\
\sigma_i &= \beta(\alpha_i+3, 2\beta_i+1, \frac{D}{H}).
\end{aligned}
$$

Therefore, if we define for every m

$$u_{j,k;j',k'} = \sum_{i=1}^{4} I_i(j,k,j',k')$$

and

$$v_{j,k;j',k'} = I_5(j,k,j',k')$$

for all j' and k' Eq.(6) can be written

$$\sum_{j,k} A_{j,k} u_{j,k;j',k'} = \sum_{j,k} \kappa^2 v_{j,k;j',k'}$$

storing the above matrices by rows as

$$L = (j-m)k_{max} + k + 1,$$

$$L' = (j'-m)k'_{max} + k' + 1,$$

we will get the following matrix equation

$$\sum_L A_L u_{L,L'} = \sum_L \kappa^2 v_{L,L'}$$

which, in vector form can be written

$$\mathbf{A} \cdot \mathbf{u} = \kappa^{\mathbf{2}} \mathbf{v}$$

the above equation is an asymmetric eigenvalue problem which can be solved by well known techniques. For $\alpha = 10$ degrees and $a = 1$ in arbitrary units, we find the lowest k for $m = 0$ to be 2.88143.

The Galerkin Method

Suppose we have a linear differential or integral operator D, defined on a domain D_x. We desire a solution $\psi(x)$ of the equation

$$D\psi(x) = p(x). \tag{7}$$

If an exact solution is too difficult to obtain, then we can approximate $\psi(x)$ in terms of a finite set of basis functions $g_n(x)$

$$\psi(x) \approx \sum_{n=1}^{N} a_n g_n(x) \tag{8}$$

where the coefficients $\mathbf{a}$ are unknowns. Since D is linear, we can substitute our expansion (8) into (7) to obtain

$$D\sum_{n=1}^{N} a_n g_n(x) = \sum_{n=1}^{N} a_n D g_n(x) \approx p(x). \tag{9}$$

Since our expansion is not exact, we are left with a residual error term

$$R(x,a) = \sum_{n=1}^{N} a_n D[g_n(x)] - p(x). \tag{10}$$

In order to specify $\mathbf{a}$ in some reasonable manner, we wish to choose $\mathbf{a}$ as to minimize $R(x,a)$ in some sense. We could choose a set of M points x_k, $k = 1, 2, ..M$, and require that $R(x,a)$ be zero at each x_k. A more general approach would be to specify a set of M weighting functions $W_k(x)$, $k = 1, 2, ..M$ and require for each k that

$$\int_{D_x} W_k(x)R(x,a)dx = \sum_{n=1}^{N} a_n \int_{D_x} W_k(x)D(g_n(x))dx - \int_{D_x} W_k(x)p(x)dx = 0 \tag{11}$$

which can be written in matrix form as

$$Y\mathbf{a} = \mathbf{b}$$

where the $Y_{k,n}$ element of the matrix Y is given

$$Y_{k,n} = \int_{D_x} W_k(x)D(g_n(x))dx$$

and the kth element of the b vector is

$$b_k = \int_{D_x} W_k(x)p(x)dx.$$

In the Galerkin method the weighting functions $W_k(x)$ are chosen to be the same as the basis functions $g_x(x)$ so

$$Y_{k,n} = \int g_k(x) D(g_n(x)) dx$$

and

$$b_k = \int g_k(x) p(x) dx$$

Note that in Dirac notation this can be written for all k and n

$$Y_{k,n} =< g_k(x)|D|g_n(x) >$$

and

$$b_k =< g_k(x)|p(x) >$$

from here Eq.(11) can be written for each k

$$\sum_{n=1}^{N} a_n < g_k(x)|D|g_n(x) >=< g_k(x)|p(x) >$$

from which a_n can be determined.