Line integral formula for scattering of waves from a thin plate

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We derive formulas for the calculation of scattering amplitude for plates of arbitrary shape. These formulas are line integrals of the coefficient of the field singularity around the edges of the plate. If this coefficient is obtained from the exact solution of the wave equation, then these formulas give the scattering amplitude exactly. If locally this coefficient is approximated by the coefficient of the field singularity from a half plane, then these formulas give the results obtained by the geometrical theory of diffraction when only singly diffracted rays are considered.

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I. INTRODUCTION

As is well known, the determination of the scattered field from an object by Huygen's principle involves an integral over the surface of the object. (According to Babinet's principle the same type of integral can be applied to determine the diffracted field from an aperture, therefore the words "scattered" and "diffracted" are used interchangeably throughout this paper.) In dealing with the problem of diffraction of light from an aperture, it was first suggested by Young that diffraction was an edge effect, but this suggestion was generally ignored. After Kirchhoff converted Huygens' wave theory into a quantitative form which represents the diffracted field as an integral over the aperture, it was shown by Maggi that this integral could be split into two parts, one of which was the field distribution predicted by geometrical optics, and the other of which was a line integral over the edge of the aperture. Later Rubinowicz [1] evaluated the line integral asymptotically for the case where the wavelength of the incident field was much smaller than the size of the aperture. His results gave an explicit expression for the diffracted field and showed that the diffracted field came from the neighborhood of the stationary phase points on the edge of the aperture. Thus Young's assertion was confirmed and was converted to a quantitative procedure for determining the diffracted field. However, all this work was done in the context of the Kirchhoff approximation. In this paper we derive an exact expression for the scattering amplitude from a flat object in terms of a line integral around its edges.

In a previous paper about the theory of scattering of waves from surfaces [2], Dashen and Wurmser derived a number of formulas which represent the scattering amplitude in terms of surfaces integrals for Dirichlet, Neumann, and electromagnetic boundary conditions. Here it is shown that when the scattering body is a plate of zero thickness the scattering indeed comes from the edges and thus these surface integrals can be reduced to line integrals around the edges of the plate. Therefore the scattering amplitude of an arbitrary planar object can be expressed as a line integral around its edges.

In this paper two-dimensional vectors are denoted by boldface variables and three-dimensional vectors are denoted by variables with an overarrow. The scattering amplitude is given by [2]

$$T_D(\vec{k},\vec{q})=\frac{i}{|\mathbf{Q}_\parallel|^2}\int_S \hat{\mathbf{n}}\cdot\mathbf{Q}_\parallel(\hat{\mathbf{n}}\cdot\nabla\psi_{\vec{k}})(\hat{\mathbf{n}}\cdot\nabla\psi_{-\vec{q}})dS \tag{1}$$

for the Dirichlet boundary condition,

$$T_N(\vec{k},\vec{q})=\frac{-i}{|\mathbf{Q}_\parallel|^2}\int_S \hat{\mathbf{n}}\cdot\mathbf{Q}_\parallel(\nabla\psi_{\vec{k}}\cdot\nabla\psi_{-\vec{q}}-k^2\psi_{\vec{k}}\psi_{-\vec{q}})dS \tag{2}$$

for the Neumann boundary condition, and

$$T_{\mathrm{EM}}(\vec{k},\vec{\epsilon}_{\vec{k}};\vec{q},\vec{\epsilon}_{\vec{q}})=\frac{-ik_0^2\mu_\infty}{|\mathbf{Q}_\parallel|^2}\int_S \hat{\mathbf{n}}\cdot\mathbf{Q}_\parallel[\epsilon\vec{E}_{\vec{k}}\cdot\vec{E}_{-\vec{q}}+\mu\vec{H}_{\vec{k}}\cdot\vec{H}_{-\vec{q}}]dS \tag{3}$$

for electromagnetic boundary conditions. In the above equations $\vec{k}$ is the incoming wave vector, $\vec{q}$ is the outgoing wave vector, $\vec{Q}=\vec{k}-\vec{q}$, $\mathbf{Q}_\parallel$ is the two-dimensional component of $\vec{Q}$ in the plane of the plate, and $\hat{\mathbf{n}}$ is the unit normal to the surface of the plate pointing into the scattering region. In the electromagnetic formula, $\vec{\epsilon}_{\vec{k}}$ and $\vec{\epsilon}_{-\vec{q}}$ are the polarization vectors.

II. THEORY

Consider a plate whose edges are parametrized by a variable s. From the above equations we see that the only contribution to the scattering amplitude comes from the edges where $\hat{\mathbf{n}}\cdot\mathbf{Q}_\parallel$ is not zero and the fields are singular. For waves subject to the Dirichlet boundary condition, the normal derivative of the field evaluated on the surface of the plate is

$$\hat{\mathbf{n}}\cdot\nabla\psi_{\vec{k}}(\vec{k},s,r,\theta)=C_D(\vec{k},s)e^{i\mathbf{x}(s)\cdot\mathbf{k}}\frac{1}{\sqrt{r}}+O(r^0)\ ,$$

which defines the function $C_D(\vec{k},s)$. In this equation r is the perpendicular distance to the edge and, for convenience, the coefficient of the singularity is written as $e^{i\mathbf{x}(s)\cdot\mathbf{k}}$ times C_D; this makes C_D independent of the coordinate system. In the above $\mathbf{x}(x)$ is a two-dimensional position vector. We get a similar equation for $\hat{\mathbf{n}}\cdot\nabla\psi_{-\vec{q}}$.

Similarly for waves subject to the Neumann boundary condition,

$$\nabla\psi_{\vec{k}}(\vec{k},s,r,\theta)=C_N(\vec{k},s)e^{i\mathbf{x}(s)\cdot\mathbf{k}}\frac{1}{\sqrt{r}}+O(r^0) ,$$

where $C_N(\vec{k},s)$ is the coefficient of the singularity near the edge. Because of the singularity of the fields on the edge, the problem needs to be regulated. To do this we deform the plate so that it ends not on a curve of zero thickness, but on a thin tube of radius r, as shown by Fig. 1. The boundary conditions on the tube are taken to be the same as on the plate. For this regulated problem all the quantities in Eqs. (1)–(3) are finite and the answer to the original problem can be recovered by taking the limit $r\to 0$. On the surface of the tube the unit normal can be written as

$$\hat{\mathbf{n}}_{\text{tube}}(s)=\cos\theta\hat{\mathbf{N}}+\sin\theta\hat{\mathbf{n}} ,$$

where $\hat{\mathbf{N}}$ is a unit vector in the plane of the plate perpendicular to and pointing away from the edge and $\hat{\mathbf{n}}$ is the original normal to the plate. The angle θ is measured with respect to $\hat{\mathbf{N}}$ in the counterclockwise direction. Then Eq. (1) can be written

$$T_D(\vec{k},\vec{q})=\lim_{r\to 0}\frac{i}{|\mathbf{Q}_\parallel|^2}\times\oint\int_{-\pi}^{\pi}\hat{\mathbf{N}}\cdot\mathbf{Q}_\parallel\cos\theta[\hat{\mathbf{n}}\cdot\nabla\psi_{\vec{k}}(\vec{k},s,r,\theta)]\times[\hat{\mathbf{n}}\cdot\nabla\psi_{-\vec{q}}(-\vec{q},s,r,\theta)]r\,d\theta\,dl(s) \quad (4)$$

Locally the normal derivatives $\hat{\mathbf{n}}\cdot\nabla\psi_{\vec{k}}$ and $\hat{\mathbf{n}}\cdot\nabla\psi_{-\vec{q}}$ have the same r and θ behavior as do the normal derivatives of the solution of the wave equation for a half plane whose edge is deformed to a tube of radius r satisfying Dirichlet boundary conditions. It can be shown that at a local point $\mathbf{x}(s)$ on the surface of the tube

$$\hat{\mathbf{n}}\cdot\nabla\psi_{\vec{k}}(\vec{k},s,r,\theta)=2C_D(\vec{k},s)e^{i\mathbf{x}(s)\cdot\mathbf{k}}\cos\frac{\theta}{2}\frac{1}{\sqrt{r}}+O(r^0)$$

and

$$\hat{\mathbf{n}}\cdot\nabla\psi_{-\vec{q}}(-\vec{q},s,r,\theta)=2C_D(-\vec{q},s)\times e^{-i\mathbf{x}(s)\cdot\mathbf{q}}\cos\frac{\theta}{2}\frac{1}{\sqrt{r}}+O(r^0)$$

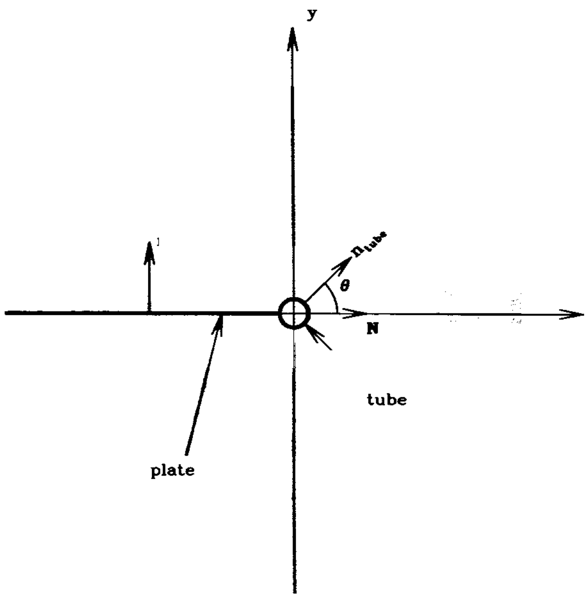

FIG. 1. Side view of a thin plate whose edge is deformed to a tube of radius r.

Substituting the above quantities in Eq. (4), carrying out the θ integration, and taking the limit yields

$$T_D(\vec{k},\vec{q})=\frac{2\pi i}{|\mathbf{Q}_\parallel|^2}\oint\hat{\mathbf{N}}\cdot\mathbf{Q}_\parallel e^{i\mathbf{x}(s)\cdot\mathbf{Q}_\parallel}C_D(\vec{k},s)C_D(\vec{q},s)dl(s) . \quad (5)$$

Let us define a unit vector $\hat{\mathbf{t}}$ in the plane of the plate and tangent to the edge such that

$$\hat{\mathbf{N}}=\hat{\mathbf{t}}\times\hat{\mathbf{n}} .$$

Then we can write

$$\begin{aligned}\hat{\mathbf{N}}\cdot\mathbf{Q}_\parallel dl(s)&=[dl(s)\hat{\mathbf{t}}\times\hat{\mathbf{n}}]\cdot\mathbf{Q}_\parallel\\&=[d\vec{l}(s)\times\hat{\mathbf{n}}]\cdot\mathbf{Q}_\parallel\\&=(\hat{\mathbf{n}}\times\mathbf{Q}_\parallel)\cdot d\vec{l}(s)=(\hat{\mathbf{n}}\times\vec{Q})\cdot d\vec{l}(s) .\end{aligned}$$

In the second line the vector identity $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$ is used. Since $|\mathbf{Q}_\parallel|^2=|\hat{\mathbf{n}}\times\vec{Q}|^2$, Eq. (5) can be written

$$T_D(\vec{k},\vec{q})=\frac{2\pi i}{|\hat{\mathbf{n}}\times\vec{Q}|^2}\int e^{i\mathbf{x}(s)\cdot\mathbf{Q}_\parallel}C_D(\vec{k},s)C_D(-\vec{q},s)\times(\hat{\mathbf{n}}\times\vec{Q})\cdot d\vec{l}(s) . \quad (6)$$

For the Neumann boundary condition

$$T_N(\vec{k},\vec{q})=\lim_{r\to 0}\frac{-i}{|\hat{\mathbf{n}}\times\vec{Q}|^2}\oint\int_{-\pi}^{\pi}\cos\theta[\nabla\psi_{\vec{k}}(\vec{k},s,r,\theta)\cdot\nabla\psi_{-\vec{q}}(-\vec{q},s,r,\theta)-k^2\psi_{\vec{k}}(\vec{k},s,r,\theta)\psi_{-\vec{q}}(-\vec{q},s,r,\theta)]r\,d\theta(\hat{\mathbf{n}}\times\vec{Q})d\vec{l}(s) , \quad (7)$$

where now

$$\nabla\psi_{\vec{k}}(\vec{k},s,r,\theta)=2C_N(\vec{k},s)e^{i\mathbf{x}(s)\cdot\mathbf{k}}\cos\frac{\theta}{2}\frac{1}{\sqrt{r}}+O(r^0)$$

and

$$\nabla\psi_{-\vec{q}}(-\vec{q},s,r,\theta)=2C_N(-\vec{q},s)e^{-i\mathbf{x}(s)\cdot\mathbf{q}}\cos\frac{\theta}{2}\frac{1}{\sqrt{r}}+O(r^0)\ .$$

Since $\psi_{\vec{k}}$ and $\psi_{-\vec{q}}$ are not singular as $r\to 0$, there is no contribution from the second term in Eq. (7) to the integral. We therefore get

$$T_N(\vec{k},\vec{q})=\frac{-2\pi i}{|\hat{\mathbf{n}}\times\vec{Q}|^2}\oint e^{i\mathbf{x}(s)\cdot\mathbf{Q}_\parallel}C_N(\vec{k},s)C_N(-\vec{q},s)(\hat{\mathbf{n}}\times\vec{Q})\cdot d\vec{l}(s)\ . \tag{8}$$

Similarly, we find that the scattering amplitude for electromagnetic waves from a plate is given by

$$T_{\mathrm{EM}}(\vec{k},\vec{\epsilon}_{\vec{k}};\vec{q},\vec{\epsilon}_{\vec{q}})=\frac{-2\pi i k_0^2\mu_\infty}{|\hat{\mathbf{n}}\times\vec{Q}|^2}\oint e^{i\mathbf{x}(s)\cdot\mathbf{Q}_\parallel}\{\epsilon C_e(\vec{k},\vec{\epsilon}_{\vec{k}},s)C_e(-\vec{q},\vec{\epsilon}_{\vec{q}},s)+\mu C_h(\vec{k},\vec{\epsilon}_{\vec{k}},s)C_h(-\vec{q},\vec{\epsilon}_{\vec{q}},s)\}(\hat{\mathbf{n}}\times\vec{Q})\cdot d\vec{l}(s)\ , \tag{9}$$

where C_e is the coefficient of $r^{-1/2}$ singularity of the normal component of $\vec{E}$ and C_h is the coefficient of $r^{-1/2}$ singularity of the tangential component of $\vec{H}$ on the edge of a thin plate.

III. CONCLUSION

It was shown in the above that scattering from a plate of zero thickness comes entirely from its edges and thus the scattering amplitude can be expressed in terms of line integrals around the edges given by Eqs. (6), (8), and (9). If the coefficients of the field singularity which appear in the integrand are obtained from an exact solution of the wave equation or Maxwell's equations, these expressions are exact. In a sense these formulas are the exact form of the Maggi-Rubinowicz formulas [3,4]. The Maggi-Rubinowicz formulas are based on the approximate Kirchoff integral which does not provide an exact description of the field [5]. Such description is obtained from the exact solution of the wave equation or Maxwell's equations plus the boundary conditions. If the wavelength of the incident field is much smaller than the size of the plate, the coefficients of the field singularity at the edge can be approximated by the coefficients of the field singularity at the edge of a half plane. In this case the scattering amplitudes obtained from these equations are equivalent to those obtained by the method of geometrical theory of diffraction [6] when only singly diffracted rays are considered. The geometrical theory of diffraction is based on the exact solution of the wave equation and, unlike the Maggi-Rubinowicz method, exhibits the correct asymptotic behavior for high frequencies [6].

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