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NUMERICAL MODELS FOR OCEAN ACOUSTIC MODES*

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We present a fast finite-difference method for computing the normal modes for three different oceanic scenarios. The first model we treat is the standard modal equation for a stratified ocean of constant depth. The ocean subbottom is modeled as a completely rigid medium and the ocean surface a perfect pressure release. This leads to a well-known eigenvalue problem of Sturm-Liouville form for the normal modes. In the second model we add to this problem a laminar shear flow parallel to the ocean bottom which yields an interesting second-order differential eigenvalue problem in which the eigenvalue appears nonlinearly. In the third model, we treat a two medium problem with the standard differential equation in the ocean coupled to a fourth-order differential equation governing elastic wave propagation in the ocean subbottom. The ocean subbottom is incorporated into the first model by a numerically computed impedance condition at the ocean-subottom interface. Finally, we present results which illustrate the speed and accuracy of the method.

1. INTRODUCTION

The method of normal modes is a standard procedure for solving acoustic propagation problems in stratified oceans. The horizontal propagation numbers k_j, $j = 1,2,\ldots$, and the corresponding depth-dependent normal modes $p_j(z)$ are then the eigenvalues and eigenfunctions, respectively, of an eigenvalue problem for an ordinary differential equation in the z-direction. The precise structure of this eigenvalue problem depends on the specific ocean acoustic problem under study and hence its corresponding mathematical model. Since the coefficients in the normal mode equation are functions of z, the eigenvalue problem is explicitly solvable only for special stratifications. In general, numerical methods are employed to solve the problem approximately. The numerical methods must solve the eigenvalue problem not just quickly but accurately, because the numerical errors in the eigenvalues appear as phase shifts in the range dependence of the acoustic field. Specifically, the phase shifts are proportional to the products of the errors and the range variable. Thus, for long-range propagation the errors in k_j will seriously degrade the normal mode representation of the solution. The speed of computation is essential, particularly for "high" frequency propagation in "deep" oceans because the number of propagating modes required to represent the acoustic field may be large.

In [1-3] we presented finite-difference numerical methods for solving the normal mode eigenvalue problems for a sequence of ocean acoustic propagation problems of increasing complexity. The numerical methods are suitably modified as the complexity of the problem increases. Reference to previous numerical studies of normal modes and the details of our numerical procedures are given in [1-3].

The standard acoustic propagation problem for a stratified, stationary ocean of uniform depth is the simplest model that we considered [1]. The ocean's surface (z=0) and bottom (z=D) were assumed to be free and rigid, respectively, where z is the depth variable. Then the normal mode eigenvalue problem for the eigenvalues k_1,

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$k_2, \ldots,$ and the corresponding eigenfunctions $p_1(z), p_2(z), \ldots$ is given by

$$p'' + [\omega^2/c^2(z) - k^2]_{;}=0 \quad , \; 0<z<D \quad ; \qquad (1.1a)$$

$$p(0)=0 \quad , \qquad (1.1b)$$

$$p'(D)=0 \quad . \qquad (1.1c)$$

Here, $c(z)$ is the specified sound speed and ω is the specified angular frequency of the time-periodic source. The numerical methods that we employed to solve (1.1) are briefly summarized in Section 2 of this paper.

If the acoustic propagation problem for the stratified ocean of uniform depth is modified by the presence of unidirectional (x-direction) laminar shear flow with velocity $U(z)$, then the normal mode eigenvalue problem is given by [2],

$$[p'/a(z)]' + b(z)p=0 \; , \; 0<z<D \; ; \qquad (1.2a)$$

$$p(0) = 0 \quad , \qquad (1.2b)$$

$$p'(D)/a(D) = 0 \quad . \qquad (1.2c)$$

The functions $a(z)$ and $b(z)$ are defined by

$$\begin{aligned} a(z) &= [\,\omega - kU(z)\,]^2 \\ b(z) &= c^{-2}(z) + k^2/a(z) \; . \end{aligned} \qquad (1.2d)$$

The problem (1.2) is not a standard Sturm-Liouville eigenvalue problem since the eigenvalue parameter k appears nonlinearly. The necessary modifications in the numerical procedures that were used to solve (1.1) are discussed in [2]. If the medium is stationary, so that $U(z) = 0$, then (1.2) is reduced to (1.1).

For "low" frequency propagation in "shallow" oceans some of the acoustic energy from the source may interact significantly with the ocean subbottom. In [3], we have modeled the ocean subbottom by a stratified elastic layer resting on a rigid half-space. This half-space corresponds to relatively rigid basement rock. Then the normal mode eigenvalue problem for this coupled seismo-acoustic model consists of simultaneously solving the acoustic normal mode equation (1.1a) with surface condition (1.1b) and the elastic normal mode equations for the four-dimensional elastic vector $\underline{r}(z)$, which we write as [4],

$$\underline{r}' = E\underline{r} \quad . \qquad (1.3)$$

Here $\underline{r}$ is the vector with components r_1, r_2, r_3, r_4, which are defined by

$$ikr_1 = u \quad , \; r_2 = w \quad ,$$

$$ikr_3 = \tau_{zx} \quad , \; r_4 = \tau_{zz} \quad , \qquad (1.4)$$

where the quantities $u(z)$, $w(z)$, $\tau_{zx}(z)$ and $\tau_{zz}(z)$ are proportional to the x-displacement, the z-displacement, the shear stress and the normal stress, respectively in the elastic layer. In addition, E is the 4x4 matrix defined by

$$E = \begin{bmatrix} 0 & -1 & 1/(\rho c_s^2) & 0 \\ k^2\eta(z) & 0 & 0 & 1/(\rho c_p^2) \\ k^2\zeta(z)-\rho\omega^2 & 0 & 0 & -\eta(z) \\ 0 & -\rho\omega^2 & k^2 & 0 \end{bmatrix} \qquad (1.4b)$$

where the quantities $\zeta(z)$ and $\eta(z)$ are defined by

$$\eta(z) = [c_p^2 - 2c_s^2]/c_p^2, \quad \zeta = c_p^2[1-\eta^2(z)] . \qquad (1.4c)$$

Here, $c_p(z)$ and $c_s(z)$ are the P and S wave speeds in the elastic bottom and ρ is its density. The system must be solved subject to the rigid conditions,

$$r_1(D_b) = r_2(D_b) = 0 \quad , \qquad (1.5a)$$

at the basement and the interfacial conditions,

$$\begin{aligned} \omega^2 r_2(D) &= p'(D) \quad , \\ r_3(D) &= 0 \; , \\ r_4(D) &= -p(D) , \end{aligned} \qquad (1.5b)$$

at the ocean bottom. The eigenvalue problem is: for specified ω, ρ, $c(z)$, $c_p(z)$, and $c_s(z)$ determine the values of k for which (1.1a,b), (1.3)-(1.5) have non-trivial solutions. The numerical methods employed in [1] to solve (1.1), require substantial modifications to solve the coupled acoustic-elastic eigenvalue problem as described in [3]. Several applications of the methods are also given in [3].

2. SUMMARY OF THE NUMERICAL METHOD FOR THE STANDARD NORMAL MODE PROBLEM (1.1)

We employ the standard three-point difference approximation to the second derivative in (1.1a) and the centered difference approximation to the first derivative in (1.1c) to reduce (1.1) to an algebraic eigenvalue problem with a tridiagonal matrix. For a fixed mesh width h_1 we then use the Sturm sequence method [5] to obtain isolating intervals for all the eigenvalues of the algebraic problem, corresponding to propagating modes. The isolating intervals provide initial estimates for each eigenvalue. More accurate approximations of each eigenvalue are then obtained by solving the characteristic equation by Brent's method [6] which combines bisection, linear interpolation and inverse quadratic interpolation. Convergence is then guaranteed to the isolated eigenvalue. Once the eigenvalues have been obtained, the corresponding eigenvectors are computed by inverse iteration.

We then employ the standard Richardson extrapolation method [7] to obtain improved estimates of the eigenvalues of (1.1) from the numerical eigenvalues of the algebraic problem. That is, the algebraic problem is solved for a sequence of successively finer meshes $h_1, h_2, \ldots, h_m$. Richardson approximations are then obtained by extrapolating the error to zero mesh width. Clearly the Richardson approximations depend on the selected values of $h_1, h_2, \ldots, h_m$. The extrapolation is based on the fact that the difference approximation employed yields an error which is a series in even powers of h. Hence, the extrapolation is in polynomials in even powers of h. Furthermore, we note that there exists a simple recursion for computing the Richardson extrapolates. Since the details of this procedure are given in [1] and references therein, we do not repeat them here.

Richardson extrapolation may also be applied to methods based on higher-order difference approximations such as the fourth-order Numerov's difference scheme. However, we have found that the lower-order difference scheme employed in [1] is more efficient as well as simpler and more easily generalized to more complicated problems. We have also experimented with modifications of the polynomial extrapolation and obtained improved results with an alternate extrapolation procedure which is described in [1].

Once the eigenvalue problem is solved at the first mesh, these eigenvalues provide excellent guesses to the eigenvalues at a refined mesh. Indeed, one may also employ extrapolation to produce an initial guess at the refined mesh and this is the procedure we have employed. The use of these coarse-mesh eigenvalues for a finer mesh yields extremely good initial guesses and allows us to bypass the root isolation procedure. For the refined meshes, this leads us to replace the Brent root finder with an alternate which does not require an isolating interval. We have employed Newton's method as the root finder for our calculations.

3. RESULTS

In [1] we applied the method to several problems to demonstrate its convergence properties, speed, accuracy and versatility. For example, we considered a deep ocean with a Munk sound speed profile [8] specified by,

$$\omega = 20\pi/s \quad , \qquad D = 5000\text{m} \quad ,$$

$$c(x) = 1500[1-0.00737(x-1+e^{-x})]\text{m/s} \quad ,$$

$$x = 2(z-1300)/1300 \; .$$

In Table 1a we present selected eigenvalues obtained at several different mesh widths given by h = D/N and with N indicated on the top row. The row labeled ET indicates the execution time in seconds on a VAX 11/750 required to compute all of the propagating modes for that particular mesh width. We have employed the notation $K_j(p)$ to denote the scaled jth eigenvalue for the mesh width h_p.

Qualitative features of the method are indicated in Table 1b in which we have displayed the errors in each of these eigenvalues. For these calculations we employed several more refined meshes to obtain reference eigenvalues which are exact to machine precision. There are two trends: the error for a fixed mesh increases with the mode number j, reflecting a greater discretization error for those modes with greater oscillation; the error decreases as the mesh width decreases. In Table 2a we have employed Richardson extrapolation with the data of Table 1a to obtain the improved approximations of the eigenvalues given in Table 2a. The first column is the same as in Table 1a. Subsequent columns represent the results of quadratic, quartic, etc. polynomial extrapolations. The effective computation time for each column is obtained by taking the sum of all the execution times for that column and all columns to its left. In Table 2a we present errors for these eigenvalues. Each subsequent extrapolation reduces the error by 2 to 3 orders of magnitude.

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Table 1a. Numerical eigenvalues $K_j^2(p) = 1000 \times k_j^2(p)$ for the Munk profile.

N=	193	231	289	366	
ET=	2.7	1.9	1.5	1.9	
j	$K_j^2(1)$	$K_j^2(2)$	$K_j^2(3)$	$K_j^2(4)$	"Exact"
1	1.7491072356	1.7491068549	1.7491065372	1.7491063255	1.7491059750
6	1.6993426880	1.6993252812	1.6993107611	1.6993010945	1.6992850968
11	1.6559087989	1.6558577581	1.6558152245	1.6557869303	1.6557401430
16	1.6114541597	1.6112937225	1.6111597366	1.6110704597	1.6109225724
21	1.5430924249	1.5426195951	1.5422242681	1.5419606260	1.5415234989
26	1.4551254442	1.4540118166	1.4530795099	1.4524571423	1.4514241517
31	1.3487793758	1.3465211555	1.3446278213	1.3433624970	1.3412598454
36	1.2249371286	1.2208224344	1.2173667722	1.2150543857	1.2112065699
41	1.0844907483	1.0775683217	1.0717434893	1.0678401041	1.0613348824
46	0.9284043428	0.9174549374	0.9082217833	0.9020243151	0.8916780490
51	0.7577294166	0.7412402724	0.7273023517	0.7179300160	0.7022535014
56	0.5736063090	0.5497490461	0.5295296282	0.5159062520	0.4930711100
61	0.3772603406	0.3438725690	0.3154938323	0.2963311179	0.2641368093
66	0.1699954891	0.1245663961	0.0858309888	0.0596128189	0.0154543453

Table 1b. Errors in numerical eigenvalues $e_j(p) = k_j^2 - k_j^2(p)$ for the Munk profile.

j	$e_j(1)$	$e_j(2)$	$e_j(3)$	$e_j(4)$
1	-1.3E-09	-8.8E-10	-5.6E-10	-3.5E-10
6	-5.8E-08	-4.0E-08	-2.6E-08	-1.6E-08
11	-1.7E-07	-1.2E-07	-7.5E-08	-4.7E-08
16	-5.3E-07	-3.7E-07	-2.4E-07	-1.5E-07
21	-1.6E-06	-1.1E-06	-7.0E-07	-4.4E-07
26	-3.7E-06	-2.6E-06	-1.7E-06	-1.0E-06
31	-7.5E-06	-5.3E-06	-3.4E-06	-2.1E-06
36	-1.4E-05	-9.6E-06	-6.2E-06	-3.8E-06
41	-2.3E-05	-1.6E-05	-1.0E-05	-6.5E-06
46	-3.7E-05	-2.6E-05	-1.7E-05	-1.0E-05
51	-5.5E-05	-3.9E-05	-2.5E-05	-1.6E-05
56	-8.1E-05	-5.7E-05	-3.6E-05	-2.3E-05
61	-1.1E-04	-8.0E-05	-5.1E-05	-3.2E-05
66	-1.5E-04	-1.1E-04	-7.0E-05	-4.4E-05

Table 2a. Standard extrapolations $K_j^2(1,\ldots,p)=1000 \times k_j^2(1,\ldots,p)$ for the Munk profile.

j	$K_j^2(1)$	$K_j^2(1,2)$	$K_j^2(1,2,3)$	$K_j^2(1,2,3,4)$	"Exact"
N=	193	231	289	366	
ET=	2.7	1.9	1.5	1.9	
1	1.7491072356	1.7491059748	1.7491059750	1.7491059750	1.7491059750
6	1.6993426880	1.6992850389	1.6992850969	1.6992850968	1.6992850968
11	1.6559087989	1.6557397577	1.6557401435	1.6557401430	1.6557401430
16	1.6114541597	1.6109228110	1.6109225751	1.6109225724	1.6109225724
21	1.5430924249	1.5415264696	1.5415235108	1.5415234989	1.5415234989
26	1.4551254442	1.4514372441	1.4514241927	1.4514241516	1.4514241517
31	1.3487793758	1.3413004226	1.3412599741	1.3412598451	1.3412598454
36	1.2249371286	1.2113097577	1.2112069378	1.2112065691	1.2112065699
41	1.0844907483	1.0615645069	1.0613358429	1.0613348810	1.0613348824
46	0.9284043428	0.8921412333	0.8916803509	0.8916780473	0.8916780490
51	0.7577294166	0.7031193603	0.7022586115	0.7022535010	0.7022535014
56	0.5736063090	0.4945939328	0.4930817189	0.4930711162	0.4930711100
61	0.3772603406	0.2666840695	0.2641575877	0.2641368357	0.2641368098
66	0.1699954891	0.0195398145	0.0154930307	0.0154544198	0.0154543458

Table 2b. Errors in standard extrapolations $e_j(1,\ldots,p) = k_j^2 - k_j^2(1,\ldots,p)$ for the Munk profile.

j	$e_j(1)$	$e_j(1,2)$	$e_j(1,2,3)$	$e_j(1,2,3,4)$
1	-1.3E-09	2.1E-13	-5.4E-17	-5.8E-18
6	-5.8E-08	5.8E-11	-5.7E-14	-6.1E-15
11	-1.7E-07	3.9E-10	-5.4E-13	4.4E-15
16	-5.3E-07	-2.4E-10	-2.7E-12	5.9E-15
21	-1.6E-06	-3.0E-09	-1.2E-11	3.4E-14
26	-3.7E-06	-1.3E-08	-4.1E-11	1.2E-13
31	-7.5E-06	-4.1E-08	-1.3E-10	3.5E-13
36	-1.4E-05	-1.0E-07	-3.7E-10	7.9E-13
41	-2.3E-05	-2.3E-07	-9.6E-10	1.4E-12
46	-3.7E-05	-4.6E-07	-2.3E-09	1.8E-12
51	-5.5E-05	-8.7E-07	-5.1E-09	4.3E-13
56	-8.1E-05	-1.5E-06	-1.1E-08	-6.2E-12
61	-1.1E-04	-2.5E-06	-2.1E-08	-2.6E-11
66	-1.5E-04	-4.1E-06	-3.9E-08	-7.4E-11